

The University of Chicago

THE MOTION OF AN UNSYMMETRIC TOP

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1935

By

EZRA JOHN CAMP

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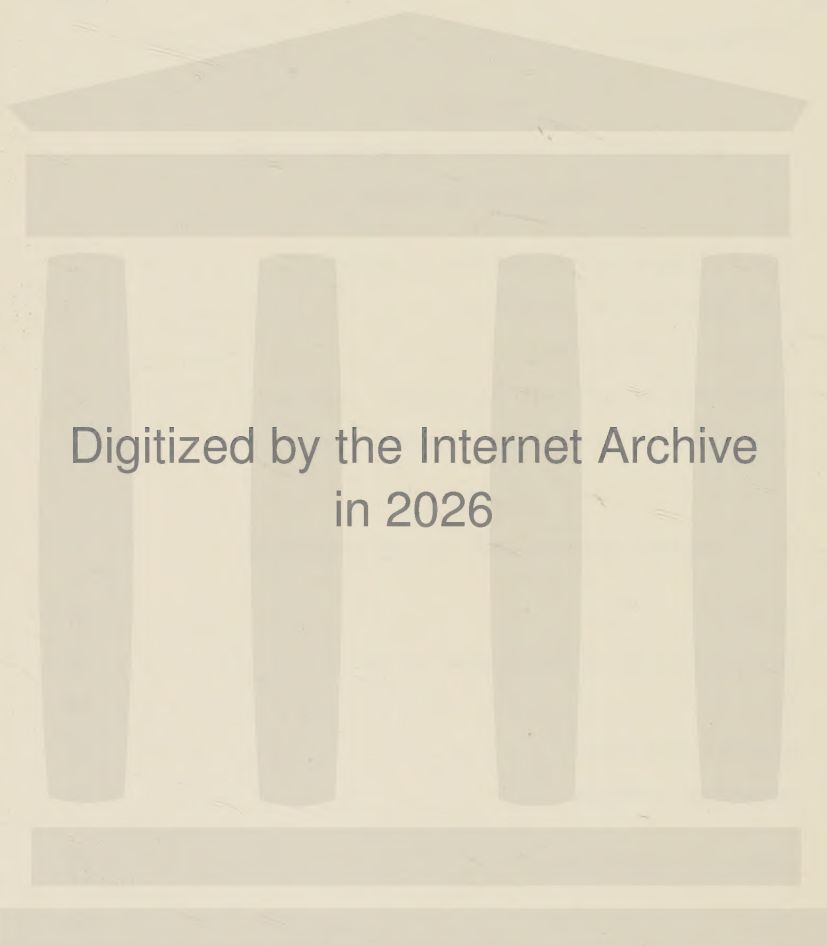
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THE MOTION OF AN UNSYMMETRIC TOP

INTRODUCTION

The problem of the motion of a rigid body about a fixed point has attracted the attention of mathematicians since the early part of the eighteenth century. Euler was among the first to obtain valuable results in the problem, the most important being the differential equations of motion which bear his name. He obtained the solution of these differential equations for the special case in which the body moves under the action of gravity with the point of support at the center of mass. Another interesting special case is that due to Lagrange in which a body, whose ellipsoid of inertia at the center of mass is a spheroid, moves under the action of gravity with the point of support taken to be any point on a principal axis of inertia through the center of mass. The complete integration of the Euler equations for the Lagrange case, in terms of the elliptic functions, was carried out recently by MacMillan [3]*.

There have been various other solutions of the Euler equations for special values of the moments of inertia and of the vertical projection of the angular momentum. A number of these solutions are outlined in a paper by J. Corliss [6]. The bibliography given by Klein and Sommerfeld [4] covers most of the important papers prior to 1900, while some of the more modern papers in the subject have been mentioned by Gray [5].

*Numbers in brackets refer to the list of references at the end of this paper.

There is an important difference in the nature of the restrictions made in this paper and those of other papers on the unsymmetric top. In most of the other papers, restrictions are made on the shape of rigid body under consideration (i.e. on the moments of inertia and the position of the center of gravity). In the present paper the only restrictions which will be imposed are the conditions on the initial values of the variables and on the nature of the constraint.

More precisely, it is the purpose of this paper to study the motion of a rigid body which moves under gravity in such a way that one of the principal axes of inertia through the center of gravity is nearly vertical, and with the further restriction that a point of this axis, not necessarily the center of gravity, remains fixed. The question of particular interest in this problem is that of the existence of periodic motion near equilibrium and the conditions under which such motion exists. By this particular type of periodic motion is meant the motion which gives rise to periodic solutions of the Euler differential equations in the neighborhood of the vertical position.

CHAPTER I

EXISTENCE OF PERIODIC SOLUTIONS OF EULER'S EQUATIONS

1. The differential equations of motion. The momental ellipsoid of a rigid body at the center of gravity G has the property that a principal axis of inertia at G is a principal axis of inertia for any of its points [3,22]. Consider one of the principal axes of inertia at the center of gravity G of an arbitrary rigid body. Let a point O of this axis be chosen as a fixed point in the body. Then the axis (OG) is one of the three principal axes of inertia at point O . Let the axis of ξ of a system of coordinates attached to the body be the line (OG) directed positively from O to G . Let the axes of η and ζ be chosen along the two remaining principal axes of inertia at point O in such a manner that the axes (ξ, η, ζ) form a right hand system. This moving system of axes will be referred to a right hand system of fixed axes with origin at point O and such that the positive direction of the axis of z is directed vertically upward. The coordinates referred to these axes will be denoted by (x, y, z) .

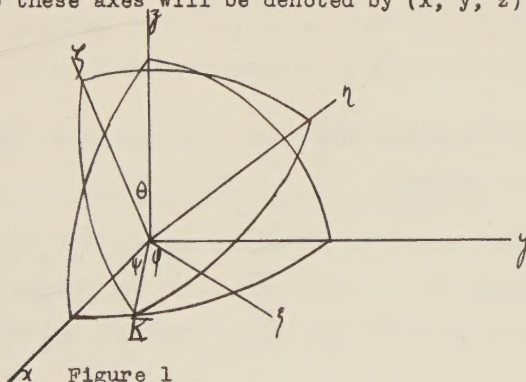


Figure 1

The angles of Euler will be defined in the following manner: let ψ be the angle which the ascending node makes with the axis of x , measured from the axis to the node counterclockwise, θ the angle from the vertical to the axis of z , and ϕ the angle from the ascending node to the axis of ξ . The moments of inertia of the body with respect to the axes (ξ, η, ζ) will be denoted by A, B , and C respectively; the components of the resultant moment of applied forces (at the center of gravity) along the moving axes by L, M , and N respectively; the components of the instantaneous angular velocity vector along these axes by $(\omega_1, \omega_2, \omega_3)$; and finally, the direction cosines of these axes with respect to the vertical by (Y_1, Y_2, Y_3) .

These direction cosines are related to the angles of Euler by the following equations [3§52]:

$$(1:1) \quad \begin{aligned} Y_1 &= \sin \phi \sin \theta, & Y_1' &= \sin \phi \cos \theta \theta' + \cos \phi \sin \theta \phi', \\ Y_2 &= \cos \phi \sin \theta, & Y_2' &= \cos \phi \cos \theta \theta' - \sin \phi \sin \theta \phi', \\ Y_3 &= \cos \theta, & Y_3' &= -\sin \theta \theta'. \end{aligned}$$

The relations between the ω_1 and the angles of Euler are given by the equations

$$(1:2) \quad \begin{aligned} \omega_1 &= \psi' \sin \phi \sin \theta + \theta' \cos \phi, \\ \omega_2 &= \psi' \cos \phi \sin \theta - \theta' \sin \phi, \\ \omega_3 &= \psi' \cos \theta + \phi'. \end{aligned}$$

On comparing equations (1:1) and (1:2) the following relations are evident:

$$(1:3) \quad \begin{aligned} Y_1' &= \omega_3 Y_2 - \omega_2 Y_3, \\ Y_2' &= \omega_1 Y_3 - \omega_3 Y_1, \\ Y_3' &= \omega_2 Y_1 - \omega_1 Y_2. \end{aligned}$$

The three equations of Euler in this notation are [5g8]:

$$(1:4) \quad \begin{aligned} A \omega_1' + (C - B) \omega_2 \omega_3 &= L, \\ B \omega_2' + (A - C) \omega_3 \omega_1 &= M, \\ C \omega_3' + (B - A) \omega_1 \omega_2 &= N. \end{aligned}$$

Under the assumption that the only force acting is that of gravity, it is easily verified that the values of (L,M,N) are

$$L = mg(\gamma_2 \zeta_0 - \gamma_3 \eta_0), \quad M = mg(\gamma_3 \xi_0 - \gamma_1 \zeta_0), \quad N = mg(\gamma_1 \eta_0 - \gamma_2 \xi_0),$$

where m is the mass and (ξ_0, η_0, ζ_0) are the coordinates of the center of gravity. But it was assumed at the outset that the fixed point O of the body should lie on a principal axis of inertia through the center of gravity and that the axis of ζ should pass through the center of gravity. It follows that ξ_0 and η_0 vanish.

Equations (1:3) and (1:4) taken together may be regarded as a sixth order system of differential equations which, by virtue of the simplifications described above, take the following form:

$$(1:5) \quad \begin{aligned} \omega_1' + m_1 \omega_2 \omega_3 &= K_1 \gamma_2, & \gamma_1' &= \gamma_2 \omega_3 - \gamma_3 \omega_2, \\ \omega_2' + m_2 \omega_3 \omega_1 &= -K_2 \gamma_1, & \gamma_2' &= \gamma_3 \omega_1 - \gamma_1 \omega_3, \\ \omega_3' + m_3 \omega_1 \omega_2 &= 0, & \gamma_3' &= \gamma_1 \omega_2 - \gamma_2 \omega_1, \end{aligned}$$

where the new constants appearing in the equations have the values:

$$(1:6) m_1 = \frac{C-B}{A}, \quad m_2 = \frac{A-C}{B}, \quad m_3 = \frac{B-A}{C}, \quad K_1 = \frac{mg\zeta_0}{A}, \quad K_2 = \frac{mg\zeta_0}{B}.$$

Equations (1:5) will furnish the basis for further discussion.

2. The equations of variation. In order to make a study of the motion under the hypothesis that θ is small, it is convenient to make a transformation of variables in the following manner:

$$\begin{aligned}
 \omega_1 &= \epsilon u_1, & \omega_2 &= \epsilon u_2, & \omega_3 &= n + \epsilon u_3, \\
 (2:1) \quad \gamma_1 &= \epsilon v_1, & \gamma_2 &= \epsilon v_2, & \gamma_3 &= 1 + \epsilon v_3, \\
 t - t_0 &= (1 + \delta) \tau.
 \end{aligned}$$

On making these substitutions in equations (1:5) and dividing through by ϵ , the transformed system becomes:

$$\begin{aligned}
 (2:2) \quad \frac{du_1}{(1+\delta)d\tau} &= -m_1 n u_2 - m_1 \epsilon u_2 u_3 + K_1 v_2, \\
 \frac{du_2}{(1+\delta)d\tau} &= -m_2 n u_1 - m_2 \epsilon u_3 u_1 - K_2 v_1, \\
 \frac{du_3}{(1+\delta)d\tau} &= -m_3 \epsilon u_1 u_2, \\
 \frac{dv_1}{(1+\delta)d\tau} &= n v_2 - u_2 + \epsilon (v_2 u_3 - v_3 u_2), \\
 \frac{dv_2}{(1+\delta)d\tau} &= u_1 - n v_1 + \epsilon (v_3 u_1 - v_1 u_3), \\
 \frac{dv_3}{(1+\delta)d\tau} &= \epsilon (v_1 u_2 - v_2 u_1).
 \end{aligned}$$

For $\epsilon = \delta = 0$ equations (2:2) form a set of linear equations with constant coefficients and the general solution is readily obtainable. For special values of the constants of integration this solution is periodic. Then the initial values of the dependent variables will be changed by small amounts and δ and ϵ will be taken different from zero. Equations (2:2) will then be integrated formally as power series in δ , ϵ , and the increments to the initial values of the dependent variables. Since the differential equations do not contain τ explicitly, these solutions will be periodic in τ with the same period as that of the solutions for $\epsilon = \delta = 0$, provided all the dependent variables take the same values after τ has traversed a period. It will be shown that the initial conditions and δ can be determined as power series in ϵ so that the conditions for periodicity

will be satisfied.

When $\delta = \epsilon = 0$ in equations (2:2) these equations reduce to:

$$(2:3) \quad \begin{aligned} \frac{du_1}{d\tau} &= -m_1nu_2 + K_1v_2, & \frac{dv_1}{d\tau} &= nv_2 - u_2, \\ \frac{du_2}{d\tau} &= -m_2nu_1 - K_2v_1, & \frac{dv_2}{d\tau} &= u_1 - nv_1, \\ u_3 &= b, & v_3 &= c, \end{aligned}$$

where b , and c are constants. In order to integrate equations (2:3) it will be convenient to reduce them to a system of two second order equations. When the equations in the second column of (2:3) are solved for u_1 and u_2 and their values substituted in the first column, there results the system of second order equations

$$(2:4) \quad \begin{aligned} \frac{d^2v_1}{d\tau^2} - n(1 + m_2)\frac{dv_2}{d\tau} - (m_2n^2 + K_2)v_1 &= 0, \\ \frac{d^2v_2}{d\tau^2} + n(1 - m_1)\frac{dv_1}{d\tau} + (m_1n^2 - K_1)v_2 &= 0. \end{aligned}$$

Equations (2:3), or the equivalent system (2:4), are called the equations of variation since their solution determines the nature of the variation of the solution of (1:5) from the particular solution $\omega_1 = 0$, $\omega_2 = 0$, $\omega_3 = \eta$, $\gamma_1 = 0$, $\gamma_2 = 0$, $\gamma_3 = 1$.

The standard method of solving equations of the type of (2:4) is to assume a solution of the form

$$v_1 = c_1 e^{\lambda\tau}, \quad v_2 = c_2 e^{\lambda\tau},$$

and to substitute these quantities in the differential equations. The differential equations will be satisfied by these functions if and only if the coefficient of $e^{\lambda\tau}$ in each equation is zero. This condition gives two linear equations in c_1 and c_2 viz.:

$$(2:5) \quad \begin{aligned} (\lambda^2 - m_2 n^2 - K_2) c_1 - n(1 + m_2) \lambda c_2 &= 0, \\ n(1 - m_1) \lambda c_1 + (\lambda^2 + m_1 n^2 - K_1) c_2 &= 0. \end{aligned}$$

A necessary and sufficient condition for equations (2:5) to have a solution for c_1 and c_2 is that the determinant vanish. This condition gives a fourth order equation in λ viz.:

$$(2:6) \quad \lambda^4 + [n^2(1 - m_1 m_2) - K_1 - K_2] \lambda^2 - m_1 m_2 n^4 + n^2(m_2 K_1 - m_1 K_2) + K_1 K_2 = 0.$$

Equation (2:6) will be referred to in the future as the characteristic equation. Its discriminant after a few simple reductions is found to be

$$(2:7) \quad D = n^4(1 + m_1 m_2)^2 + n^2[-2(1 - m_1 m_2)(K_1 + K_2) + 4(m_1 K_2 - m_2 K_1)] + (K_1 - K_2)^2.$$

3. Stability of motion. The motion under consideration will be said to be stable when all of the roots of the characteristic equation of the form of (2:6) are unequal and their real parts are either zero or negative. The following lemma will be established:

LEMMA 3:1. A necessary and sufficient condition that the motion of an unsymmetric top be stable in the neighborhood of the vertical position is that all of the roots of the characteristic equation be pure imaginaries.

The condition is sufficient by definition. That it is necessary follows from the fact that the two values of λ^2 satisfying equation (2:6) are either real or conjugate imaginary. If they are real then the four corresponding values of λ are either pure imaginary, or at least one of them is real and positive and the motion is not stable. Conjugate imaginary values

for λ^2 yield at least one value for λ which will have its real part positive, contrary to hypothesis. Hence if the motion is stable all of the roots are pure imaginaries.

In studying the characteristic equation (2:6) and its discriminant (2:7) it will be convenient to make the following transformations:

$$(3:1) \quad n = [(mg J_0)^{1/2} \eta] / (C)^{1/2}, \quad \lambda = [(mg J_0)^{1/2} x] / (C)^{1/2},$$

$$\frac{C}{A} = \mu_1, \quad \frac{C}{B} = \mu_2.$$

In view of identities (1:6), the relations (3:1) imply

$$m_1 = \mu_1(1 - \frac{1}{\mu_2}), \quad m_2 = \mu_2(\frac{1}{\mu_1} - 1), \quad K_1 = \frac{mg J_0 \mu_1}{C},$$

$$K_2 = \frac{mg J_0 \mu_2}{C}.$$

On substituting these values in equation (2:6), this equation and its discriminant reduce to the following:

$$x^4 + [\eta^2(2 + \mu_1\mu_2 - \mu_1 - \mu_2) - \mu_1 - \mu_2]x^2$$

$$+ (1 + \mu_1\mu_2 - \mu_1 - \mu_2)\eta^4 + (\mu_1 + \mu_2 - 2\mu_1\mu_2)\eta^2$$

$$+ \mu_1\mu_2 = 0,$$

$$(3:2) \quad D = \eta^4(\mu_1 + \mu_2 - \mu_1\mu_2)^2 + 2\eta^2(\mu_1 + \mu_2 - \mu_1\mu_2)(\mu_1 + \mu_2 - 4)$$

$$+ (\mu_1 - \mu_2)^2.$$

The μ_1 and μ_2 appearing in equations (3:2) are restricted by the properties of the momental ellipsoid expressed in the following inequalities:

$$A + B > C, \quad B + C > A, \quad C + A > B.$$

Expressed in terms of μ_1 and μ_2 , these conditions are:

$$(3:3) \quad \begin{aligned} \mu_1 + \mu_2 &> \mu_1 \mu_2, \quad \mu_1 - \mu_2 > -\mu_1 \mu_2, \\ \mu_2 - \mu_1 &> -\mu_1 \mu_2. \end{aligned}$$

If the inequalities in (3:3) are replaced by equalities, the resulting expressions are the equations for equilateral hyperbolas. One branch of each of these hyperbolas is represented in Figure 2 by (jkl), (abo), and (ofh) respectively. Since μ_1 and μ_2 are positive, it is necessary to consider only that part of each hyperbola which lies in the first quadrant. It is evident then that the inequalities (3:3) are satisfied for any value (μ_1, μ_2) lying in the shaded region bounded by the three hyperbolas (Figure 2). For values of μ_1 and μ_2 on the boundary of this region the body reduces to a disk.

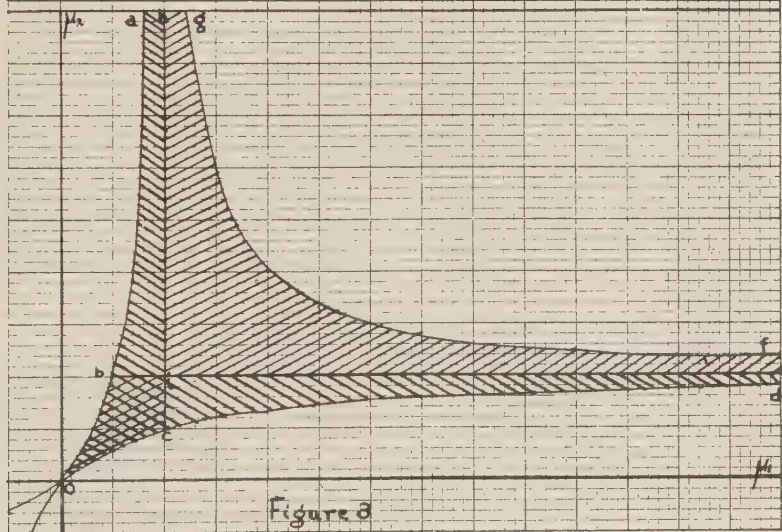
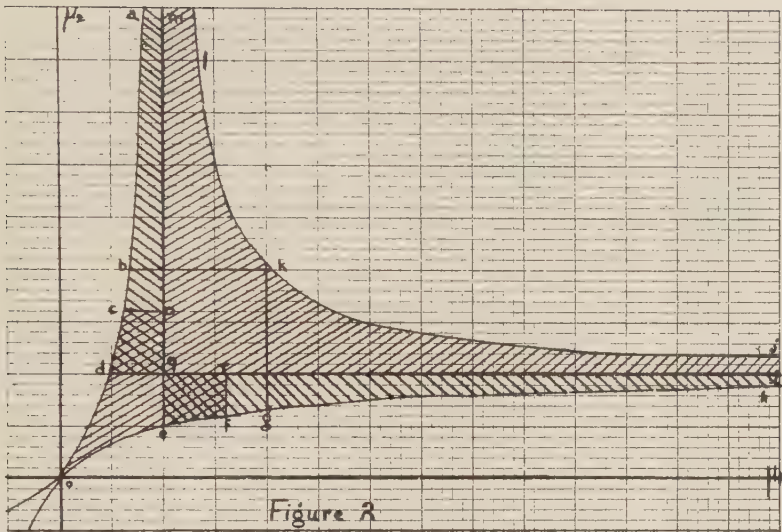
The conditions under which the quantity D of equations (3:2) is positive are given in the following lemma:

LEMMA 3:2. The quantity D of equations (3:2) is positive for values (μ_1, μ_2) within the shaded region of Figure 2 defined by the inequalities (3:3), under the following conditions:

1. $\mu_1 > 2$ or $\mu_2 > 2$,
2. $\mu_2 = 2$ or $\mu_1 = 2$, and $\eta^2 \neq 1$,
3. $\mu_2 \leq 2$, $\mu_1 \leq 2$, and $\eta^2 > \frac{2(2-\mu_1)^{1/2}(2-\mu_2)^{1/2} + 4}{\mu_1 + \mu_2 - \mu_1 \mu_2} - \frac{(\mu_1 + \mu_2)}{\mu_1 + \mu_2 - \mu_1 \mu_2}$.

In view of the fact that one of the μ 's must be less than 2 if the other is greater than 2, the first and second case is made evident by rewriting D in the form

$$(3:4) \quad D = [\eta^2(\mu_1 + \mu_2 - \mu_1 \mu_2) + \mu_1 + \mu_2 - 4]^2 + 4(\mu_1 - 2)(2 - \mu_2).$$



The third case is also derived from (3:4) by determining η^2 such that $D > 0$. It is permissible to divide both sides of the inequality by the coefficient of η^2 without changing the direction of the inequality, by virtue of the first relation of (3:3).

LEMMA 3:3. The necessary and sufficient conditions for stable motion in the neighborhood of the vertical position with the center of gravity above the point of support are

1. $D > 0$,
2. $\eta^2 [1 + (\mu_1 - 1)(\mu_2 - 1)] - \mu_1 - \mu_2 > 0$,
3. $[\eta^2(1 - \mu_2) + \mu_2][\eta^2(1 - \mu_1) + \mu_1] > 0$.

These are precisely the conditions that the two roots of a quadratic equation be negative, applied to the first equation of (3:2), this equation being regarded as a quadratic in x^2 . It follows then that all of the values of x which satisfy (3:2) are pure imaginary. The conclusion of the lemma then follows from lemma (3:1).

THEOREM (3:1). If a rigid body, whose motion satisfies the hypotheses of section 1, rotates about the axis of ζ directed vertically upward, the motion is stable or unstable for values of (μ_1, μ_2) in each of the regions of Figure 2 under the following conditions:

1. Region (mqijkl), motion is stable for $\mu_2 > \mu_1$ if $\eta^2 > \frac{\mu_1}{\mu_1 - 1}$; motion is stable for $\mu_1 > \mu_2$ if $\eta^2 > \frac{\mu_2}{\mu_2 - 1}$; for other values of η^2 the motion is unstable.
2. Region (doeq), motion is stable if $\eta^2 > \frac{2[2 - \mu_1](2 - \mu_2)]^{1/2} + 4 - (\mu_1 + \mu_2)}{\mu_1 + \mu_2 - \mu_1\mu_2}$, otherwise it is unstable.
3. (a) Region (cdqp), motion is stable if -

$$\frac{\mu_2}{\mu_2 - 1} > \eta^2 > \frac{2[(2 - \mu_1)(2 - \mu_2)]^{1/2} + 4 - (\mu_1 + \mu_2)}{\mu_1 + \mu_2 - \mu_1 \mu_2},$$

$$\eta^2 > \frac{\mu_1 + \mu_2}{1 + (\mu_1 - 1)(\mu_2 - 1)}, \text{ otherwise it is unstable.}$$

3. (b) Region (qefr), same conditions as in 3(a) with the role of μ_1 and μ_2 interchanged.
4. Regions (acpm) and (fhir), motion is unstable for all values of η .

The proof of this theorem is made by showing that the three conditions of lemma (3:3) are satisfied for each of the four cases of the theorem. For case 1 the first condition of lemma (3:3) is satisfied by virtue of lemma (3:2) and the fact that

$$(3:5) \quad \frac{\mu_1}{\mu_1 - 1} \geq \frac{2[(2 - \mu_1)(2 - \mu_2)]^{1/2} + 4 - (\mu_1 + \mu_2)}{\mu_1 + \mu_2 - \mu_1 \mu_2}$$

for $2 \geq \mu_1 > 1$, $2 \geq \mu_2 > 1$, and $\mu_2 \geq \mu_1$. The latter statement is proved by making the transformation

$$(3:6) \quad \Theta_1 = \mu_1 - 1, \quad \Theta_2 = \mu_2 - 1,$$

after which (3:5) reduces immediately to the equivalent expression

$$[(1 - \Theta_1)^{1/2} - \Theta_1(1 - \Theta_2)^{1/2}]^2 \geq 0.$$

It is important to notice that the denominator of the right member of (3:5) is positive by the first inequality of (3:3).

It is readily verified that the second condition of the lemma is satisfied for $\mu_1 > 1$, $\mu_2 > 1$, and $\mu_2 \geq \mu_1$ since the relation

$$(3:7) \quad \frac{\mu_1}{\mu_1 - 1} \geq \frac{\mu_1 + \mu_2}{1 + (\mu_2 - 1)(\mu_1 - 1)}$$

reduces to the equivalent relation

$$1 - \Theta_1 - \Theta_1^2(1 - \Theta_2) \geq 0 \quad (\Theta_2 \geq \Theta_1, 0 < \Theta_1 \leq 1),$$

on using the transformation (3:6). The denominator of the right member of (3:7) is positive for all values of (μ_1, μ_2) satisfying the inequalities (3:3). The third condition of the lemma holds for $\mu_2 \geq \mu_1$ in the region specified since

$$\frac{\mu_1}{\mu_1 - 1} \geq \frac{\mu_2}{\mu_2 - 1}.$$

Hence case 1 of the theorem is proved for $\mu_2 \geq \mu_1$. The proof for $\mu_1 > \mu_2$ follows by interchanging μ_1 and μ_2 throughout.

There is no difficulty in verifying that the three conditions of lemma (3:3) are satisfied for $(\mu_1 \leq 1, \mu_2 \leq 1)$. This establishes case 2 of the theorem.

Case 3 differs from the first two in that η^2 has an upper bound as well as a lower bound. The upper bound is derived from the third condition of the lemma while the two lower bounds are derived from conditions 1 and 2. Neither of the lower bounds for η^2 in case 3 dominates the other over the entire region (cdqp) or (qefr) and hence both must be stated.

The regions for case 4 are those regions for which the upper bound for η^2 in case 3 is less than or equal to one of the lower bounds and hence there is no value of η^2 which lies between the two. The dividing line between the regions (cdqp) and (adqm) is determined graphically by assigning values to μ_1 corresponding to points within the region adqm and determining the corresponding values of μ_2 such that the upper bound and the greatest lower bound for η^2 are equal. This line is not exactly parallel to the horizontal axis, as it appears in Figure 2, but it is very

nearly so.

It is possible to obtain a much stronger theorem on the stability of the motion about the axis of $\mathfrak{J}$ directed vertically downward, as one might anticipate. The only change in the transformation (2:1) will be in the $\mathfrak{Y}_3$ which will be changed to $\mathfrak{Y}_3 = -1 + \epsilon v_3$. After making the corresponding changes in equations (2:2) it is found that the new equations corresponding to (2:4) are

$$(3:8) \quad \begin{aligned} \frac{d^2 v_1}{d\tau^2} - n(1 + m_2) \frac{dv_2}{d\tau} - (m_2 n^2 - K_2) v_1 &= 0, \\ \frac{d^2 v_2}{d\tau^2} + n(1 - m_1) \frac{dv_1}{d\tau} + (K_1 + m_1 n^2) v_2 &= 0. \end{aligned}$$

The only difference between equations (3:8) and (2:4) is that the signs before K_2 and K_1 have been changed. Hence the characteristic equation for (3:8) is obtained readily from (2:6) by replacing K_2 and K_1 by their negatives in that equation. After making the transformation (3:1) the characteristic equation and its discriminant are

$$(3:9) \quad \begin{aligned} x^4 + \left\{ \eta^2 [2 + \mu_1 \mu_2 - (\mu_1 + \mu_2)] + \mu_1 + \mu_2 \right\} x^2 + [1 + \mu_1 \mu_2 - (\mu_1 + \mu_2)] \eta^4 \\ + [2\mu_1 \mu_2 - (\mu_1 + \mu_2)] \eta^2 + \mu_1 \mu_2 = 0, \\ D = \eta^4 (\mu_1 + \mu_2 - \mu_1 \mu_2)^2 + 2\eta^2 (\mu_1 + \mu_2 - \mu_1 \mu_2) [4 - (\mu_1 + \mu_2)] \\ + (\mu_1 - \mu_2)^2. \end{aligned}$$

LEMMA 3:4. The quantity D of equations (3:9) is positive for all values of (μ_1, μ_2) within the shaded region of Figure 3, defined by the inequalities (3:3).

The lemma will first be proved for $\mu_2 > 1$. On collecting the terms of μ_1^2 and μ_1 and completing the square in μ_1 ,

D has the form

$$(3:10) \quad D = \left[\eta^2(1-\mu_2)-1 \right] \mu_1 + \eta^2 \mu_2 + 4 - \mu_2 + 2 \frac{(2-\mu_2)}{\eta^2(1-\mu_2)-1} \Big]^2 \\ + 4\eta^2(\mu_2 - 2)^2 [\eta^2 + 1] \mu_2 - \eta^2 \Big] .$$

It is evident that the second term of the above expression is positive if $\mu_2 \neq 2$ and $\mu_2 > 1$. It is seen by inspection of (3:10) that if $\mu_2 = 2$ then $D > 0$ unless also $\mu_1 = 2$. But the point (2, 2) is on the boundary of the region defined by (3:3). Hence $D > 0$ for all values of (μ_1, μ_2) within the region defined by the inequalities (3:3) such that $\mu_2 > 1$. A similar argument shows that $D > 0$ for all (μ_1, μ_2) within the specified region such that $\mu_1 > 1$. It remains to prove the lemma for $\mu_1 \leq 1$, $\mu_2 \leq 1$. This is done by examining the second equation of (3:9). Since the first factor in the coefficient of η^2 is positive by the first inequality of (3:3) and the second factor is positive for $\mu_1 \leq 1$ and $\mu_2 \leq 1$, it follows that D is positive. Hence the lemma is proved.

LEMMA 3:5. The necessary and sufficient condition for stable motion in the neighborhood of the vertical position with the center of gravity below the point of support is

$$[\eta^2(1 - \mu_2) - \mu_2][\eta^2(1 - \mu_1) - \mu_1] > 0 .$$

Since the lemma refers to the motion of a physical body it is tacitly assumed that the (μ_1, μ_2) satisfy the inequalities (3:3). The condition expressed in the lemma is that the term independent of x in the first equation of (3:9) be positive. The coefficient of x^2 in this same equation is positive since

$$2 + \mu_1 \mu_2 - (\mu_1 + \mu_2) > 0 .$$

To prove the latter statement it is put in the form

$$1 - [\mu_1(1 - \frac{1}{\mu_2})][\mu_2(\frac{1}{\mu_1} - 1)] > 0.$$

This statement is true since each of the factors in brackets is numerically less than unity by virtue of the inequalities (3:3). Finally $D > 0$ by lemma (3:4). Hence the first equation of (3:9), considered as a quadratic in x^2 , has all of its roots negative. Therefore every x satisfying the equation is pure imaginary and the motion is stable. The converse follows immediately by re-tracing the steps.

THEOREM 3:2. If a rigid body, whose motion satisfies the hypotheses of section 1, rotates about the axis of $\mathcal{J}$ directed vertically downward, the motion is stable or unstable for values of (μ_1, μ_2) in each of the regions of Figure 3 under the following conditions:

1. $\mu_1 \geq 1, \mu_2 \geq 1$, motion is stable for all η^2 .
2. $\mu_1 \leq 1, \mu_2 \leq 1, \mu_2 > \mu_1$, motion is stable if $\eta^2 > \frac{\mu_2}{1 - \mu_2}$ or $\eta^2 < \frac{\mu_1}{1 - \mu_1}$; motion is unstable if $\frac{\mu_1}{1 - \mu_1} < \eta^2 < \frac{\mu_2}{1 - \mu_2}$.
3. (a) $\mu_1 < 1, \mu_2 > 1$, motion is stable if $\eta^2 < \frac{\mu_1}{1 - \mu_1}$, unstable if $\eta^2 \geq \frac{\mu_2}{1 - \mu_2}$.
3. (b) The same condition with μ_1 and μ_2 interchanged throughout.

The criterion for stability is given by lemma (3:5). In each of the three cases the motion is stable if η^2 is such that the condition in lemma (3:5) is satisfied, otherwise it is unstable.

4. Existence of periodic solutions, case 1. Let the roots of equation (2:6) be represented by

$$\lambda_1 = \alpha i, \quad \lambda_2 = -\alpha i, \quad \lambda_3 = \beta, \quad \lambda_4 = -\beta,$$

where α and β are positive real numbers and $i = (-1)^{1/2}$. On substituting these values in equations (2:5) the corresponding values for the constants are

$$(4:2) \quad \begin{aligned} c_1^{(1)} &= h_1 i c_2^{(1)}, \quad c_1^{(2)} = -h_1 i c_2^{(2)}, \quad c_1^{(3)} = h_2 c_2^{(3)}, \\ c_1^{(4)} &= -h_2 c_2^{(4)} \end{aligned}$$

where

$$\begin{aligned} h_1 &= \frac{n(1+m_2)\alpha}{-\alpha^2 - m_2 n^2 - K_2}, \quad \text{or} \quad h_1 = \frac{-\alpha^2 + m_1 n^2 - K_1}{n(1-m_1)\alpha}, \\ h_2 &= \frac{n(1+m_2)\beta}{\beta^2 - m_2 n^2 - K_2}, \quad \text{or} \quad h_2 = -\frac{(\beta^2 + m_1 n^2 - K_1)}{n(1-m_1)\beta}. \end{aligned}$$

By the identities (1:6) and the inequalities preceding (3:3) it is evident that $|m_1| < 1$ and $|m_2| < 1$ and hence the h 's are always defined.

The general solution of equations (2:3) in this notation is:

$$(4:3) \quad \begin{aligned} v_1 &= h_1 i c_2^{(1)} e^{\alpha i \tau} - h_1 i c_2^{(2)} e^{-\alpha i \tau} + h_2 c_2^{(3)} e^{\beta \tau} - h_2 c_2^{(4)} e^{-\beta \tau}, \\ v_2 &= c_2^{(1)} e^{\alpha i \tau} + c_2^{(2)} e^{-\alpha i \tau} + c_2^{(3)} e^{\beta \tau} + c_2^{(4)} e^{-\beta \tau}, \\ u_1 &= (n h_1 + \alpha) i c_2^{(1)} e^{\alpha i \tau} - (n h_1 + \alpha) i c_2^{(2)} e^{-\alpha i \tau} \\ &\quad + (n h_2 + \beta) c_2^{(3)} e^{\beta \tau} - (n h_2 + \beta) c_2^{(4)} e^{-\beta \tau}, \\ u_2 &= (n + \alpha h_1) c_2^{(1)} e^{\alpha i \tau} + (n + \alpha h_1) c_2^{(2)} e^{-\alpha i \tau} \\ &\quad + (n - \beta h_2) c_2^{(3)} e^{\beta \tau} + (n - \beta h_2) c_2^{(4)} e^{-\beta \tau}. \end{aligned}$$

The determinant of the coefficients of the $c_2^{(i)}$ for $\tau = 0$ is different from zero, for it has the value

$$= 4i(\alpha h_1 + \beta h_2)(h_2 \alpha - h_1 \beta),$$

and it is readily verified that each of the factors has a value different from zero. Their values are:

$$\alpha h_1 + \beta h_2 = \frac{-(\alpha^2 + \beta^2)}{n(1 - m_1)} \neq 0,$$

$$\alpha h_2 - \beta h_1 = \frac{-(\alpha^2 + \beta^2)n(1 + m_2)}{(\beta^2 - m_2 n^2 - K_2)(-\alpha^2 - m_2 n^2 - K_2)} \neq 0.$$

Neither of the factors in the denominator of the second equation can vanish for if they did it is evident on examining the determinant of equation (2:5) that the value of α or β which made them vanish would not satisfy equation (2:6).

Since it is desired that (4:3) be periodic it is necessary that $c_2^{(3)} = c_2^{(4)} = 0$. This condition is also sufficient. Furthermore, since τ contains the arbitrary constant t_0 there is no loss of generality in supposing that $v_1 = 0$ when $\tau = 0$. If $v_2 = a$ at time $\tau = 0$ then the remaining constants of integration have the value $c_2^{(1)} = c_2^{(2)} = a/2$ and the solution (4:3) reduces to

$$(4:4) \quad \begin{aligned} v_1 &= -ah_1 \sin \alpha \tau, & u_1 &= -(nh_1 + \alpha)a \sin \alpha \tau, \\ v_2 &= a \cos \alpha \tau, & u_2 &= (n + \alpha h_1)a \cos \alpha \tau, \\ v_3 &= c, & u_3 &= b. \end{aligned}$$

This solution is periodic with period $(2\pi)/\alpha$ and is the solution of equations (2:2) for $\epsilon = \delta = 0$. It is desired to determine the initial conditions and δ in such a way that the solution of equations (2:2) will have the same period in τ for $\epsilon \neq 0$.

It is convenient at this point to transform equations (2:2) to canonical form by the transformation of determinant Δ viz:

$$\begin{aligned}
 v_1 &= h_1 i(y_1 - y_2) + h_2(y_3 - y_4) , \\
 v_2 &= y_1 + y_2 + y_3 + y_4 , \\
 u_1 &= (nh_1 + \alpha)i(y_1 - y_2) + (nh_2 + \beta)(y_3 - y_4) , \\
 u_2 &= (n + \alpha h_1)(y_1 + y_2) + (n - \beta h_2)(y_3 + y_4) .
 \end{aligned}
 \tag{4:5}$$

If this transformation is substituted in equations (2:2) and the resulting equations are solved for the derivatives of the new variables, the transformed system has the form:

$$\begin{aligned}
 \frac{dy_1}{d\tau} &= (1 + \delta) \alpha i y_1 + (1 + \delta) \frac{(R_2 i - R_1)}{2} \epsilon , \\
 \frac{dy_2}{d\tau} &= - (1 + \delta) \alpha i y_2 - (1 + \delta) \frac{(R_2 i + R_1)}{2} \epsilon , \\
 \frac{dy_3}{d\tau} &= (1 + \delta) \beta y_3 + (1 + \delta) \frac{(R_3 + R_4)}{2} \epsilon , \\
 \frac{dy_4}{d\tau} &= - (1 + \delta) \beta y_4 + (1 + \delta) \frac{(R_3 - R_4)}{2} \epsilon , \\
 \frac{du_3}{d\tau} &= - m_3 (1 + \delta) u_1 u_2 \epsilon , \\
 \frac{dv_3}{d\tau} &= (1 + \delta) (v_1 u_2 - v_2 u_1) \epsilon ,
 \end{aligned}
 \tag{4:6}$$

where the $R_{(j)}$ have the values

$$\begin{aligned}
 R_1 &= (p_1 u_3 + q_1 v_3)(y_1 - y_2) i + (r_1 u_3 + s_1 v_3)(y_3 - y_4) , \\
 R_2 &= (p_2 u_3 - q_2 v_3)(y_1 + y_2) + (r_2 u_3 - s_2 v_3)(y_3 + y_4) , \\
 R_3 &= (p_3 u_3 + q_3 v_3)(y_1 - y_2) i + (r_3 u_3 + s_3 v_3)(y_3 - y_4) , \\
 R_4 &= (p_4 u_3 - q_4 v_3)(y_1 + y_2) + (r_4 u_3 - s_4 v_3)(y_3 + y_4) ,
 \end{aligned}
 \tag{4:7}$$

the p_j , q_j , r_j and s_j having the following values:

$$\begin{aligned}
p_1 &= \frac{m_2(nh_1+\alpha)-(n-\beta h_2)h_1}{\alpha h_1 + \beta h_2}, & q_1 &= \frac{(n-\beta h_2)(nh_1+\alpha)}{\alpha h_1 + \beta h_2}, \\
r_1 &= \frac{m_2(nh_2+\beta)-(n-\beta h_2)h_2}{\alpha h_1 + \beta h_2}, & s_1 &= \frac{(n-\beta h_2)(nh_2+\beta)}{\alpha h_1 + \beta h_2}, \\
p_2 &= \frac{h_2 m_1(n+\alpha h_1)+nh_2+\beta}{\alpha h_2 - \beta h_1}, & q_2 &= \frac{(nh_2+\beta)(n+\alpha h_1)}{\alpha h_2 - \beta h_1}, \\
r_2 &= \frac{h_2 m_1(n-\beta h_2)+nh_2+\beta}{\alpha h_2 - \beta h_1}, & s_2 &= \frac{(nh_2+\beta)(n-\beta h_2)}{\alpha h_2 - \beta h_1}, \\
p_3 &= \frac{m_2(nh_1+\alpha)-(n+\alpha h_1)h_1}{\alpha h_1 + \beta h_2}, & q_3 &= \frac{(n+\alpha h_1)(nh_1+\alpha)}{\alpha h_1 + \beta h_2}, \\
r_3 &= \frac{m_2(nh_2+\beta)-(n+\alpha h_1)h_2}{\alpha h_1 + \beta h_2}, & s_3 &= \frac{(n+\alpha h_1)(nh_2+\beta)}{\alpha h_1 + \beta h_2}, \\
p_4 &= \frac{m_1 h_1(n+\alpha h_1)+(nh_1+\alpha)}{\alpha h_2 - \beta h_1}, & q_4 &= \frac{(nh_1+\alpha)(n+\alpha h_1)}{\alpha h_2 - \beta h_1}, \\
r_4 &= \frac{m_1 h_1(n-\beta h_2)+nh_1+\alpha}{\alpha h_2 - \beta h_1}, & s_4 &= \frac{(nh_1+\alpha)(n-\beta h_2)}{\alpha h_2 - \beta h_1}.
\end{aligned}$$

Let the initial values of the dependent variables be

$$(4:8) \quad y_1 = \frac{a}{2} + \varpi_1, \quad y_2 = \frac{a}{2} + \varpi_2, \quad y_3 = \varpi_3, \quad y_4 = \varpi_4, \quad u_3 = b + \rho, \quad v_3 = c + \gamma,$$

it being understood that the ϖ_j , ρ , and γ all vanish for $\epsilon = 0$.

The general solution of equations (4:6) is of the form [2§31]:

$$\begin{aligned}
(4:9) \quad y_1 &= \left(\frac{a}{2} + \varpi_1\right) e^{\alpha(1+\delta)\tau} + \epsilon P_1(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \rho, \gamma, \delta, \epsilon; \tau), \\
y_2 &= \left(\frac{a}{2} + \varpi_2\right) e^{-\alpha(1+\delta)\tau} + \epsilon P_2(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \rho, \gamma, \delta, \epsilon; \tau), \\
y_3 &= \varpi_3 e^{\alpha(1+\delta)\tau} + \epsilon P_3(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \rho, \gamma, \delta, \epsilon; \tau), \\
y_4 &= \varpi_4 e^{-\beta(1+\delta)\tau} + \epsilon P_4(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \rho, \gamma, \delta, \epsilon; \tau), \\
u_3 &= b + \rho + \epsilon P_5(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \rho, \gamma, \delta, \epsilon; \tau), \\
v_3 &= c + \gamma + \epsilon P_6(\varpi_1, \varpi_2, \varpi_3, \varpi_4, \rho, \gamma, \delta, \epsilon; \tau),
\end{aligned}$$

where the $P_j (j=1\dots 6)$ are power series in the parameters exhibited.

Since τ does not appear in the differential equations (4:6) explicitly, sufficient conditions that the solution (4:9) shall be periodic in τ with the period $(2\pi)/\alpha$ are

$$(4:10) \quad y_j(2\pi/\alpha) - y_j(0) = 0 \quad (j=1\dots 4), \quad u_3(2\pi/\alpha) - u_3(0) = 0, \quad v_3(2\pi/\alpha) - v_3(0) = 0.$$

These conditions are not all independent since the energy integral and the identity relation between the Υ 's establish relations between them of the form

$$(4:11) \quad \begin{aligned} A\epsilon u_1^2 + B\epsilon u_2^2 + C\epsilon u_3^2 + 2Cnu_3 + 2mgJ_0v_3 - T_0 &= 0, \\ \epsilon v_1^2 + \epsilon v_2^2 + \epsilon v_3^2 + 2v_3 &= 0. \end{aligned}$$

The first of these equations is derived from the first three equations of (2:2) by multiplying them by u_1 , u_2 , and u_3 , adding, and then integrating the resulting expression. The second equation is derived from the well known relation between the direction cosines and the transformation (2:1). It is desired to prove the following

LEMMA 4:1. If the y_1 , y_2 , y_3 , and y_4 of solution (4:9) of differential equations (4:6) repeat their values after a period $(2\pi/\omega)$, then u_3 and v_3 also repeat their values after a period. Hence the first four equations of (4:10) are sufficient to insure periodicity.

In proving this lemma the following transformation of variables will be useful:

$$(4:12) \quad \begin{aligned} y_1 &= \left(\frac{a}{2} + \Upsilon_1\right)e^{\omega_1\tau} + w_1, & y_3 &= \Upsilon_3 + w_3, \\ y_2 &= \left(\frac{a}{2} + \Upsilon_2\right)e^{-\omega_1\tau} + w_2, & y_4 &= \Upsilon_4 + w_4, \\ u_3 &= b + w_5, & v_3 &= c + w_6. \end{aligned}$$

Since b and c are arbitrary¹ it may be assumed for present purposes that they absorb the ρ and Υ . After the values of u_1 , u_2 , u_3 , v_1 , v_2 , and v_3 given by the product of transformations (4:5) and (4:12) are substituted in equations (4:11) the resulting equations after subtracting the constant term are of the form

¹The quantity c can be determined as a unique power series in ϵ so that the sum of the squares of Υ_1 , Υ_2 , and Υ_3 is unity.

$$\begin{aligned}
 (4:13) \quad & F_1 \left[\left(\frac{a}{2} + \tau_1 \right) e^{\kappa i \tau + w_1}, \left(\frac{a}{2} + \tau_2 \right) e^{-\kappa i \tau + w_2}, \tau_3 + w_3, \tau_4 + w_4, b + w_5, c + w_6 \right] \\
 & - F_1 \left[\frac{a}{2} + \tau_1, \frac{a}{2} + \tau_2, \tau_3, \tau_4, b, c \right] = 0, \\
 & F_2 \left[\left(\frac{a}{2} + \tau_1 \right) e^{\kappa i \tau + w_1}, \left(\frac{a}{2} + \tau_2 \right) e^{-\kappa i \tau + w_2}, \tau_3 + w_3, \tau_4 + w_4, b + w_5, c + w_6 \right] \\
 & - F_2 \left[\frac{a}{2} + \tau_1, \frac{a}{2} + \tau_2, \tau_3, \tau_4, b, c \right] = 0.
 \end{aligned}$$

These equations have a solution $w_j = 0$ ($j = 1, \dots, 6$) at $\tau = 2\pi/\kappa$. Furthermore, the Jacobian of these functions with respect to w_5 and w_6 evaluated for $w_j = 0$ is

$$\begin{vmatrix} \frac{\partial F_1}{\partial w_5} & \frac{\partial F_1}{\partial w_6} \\ \frac{\partial F_2}{\partial w_5} & \frac{\partial F_2}{\partial w_6} \end{vmatrix} = 4C \begin{vmatrix} b+n & mg\mathfrak{J}_0 \\ 0 & \epsilon c+1 \end{vmatrix}.$$

The value of this determinant is evidently different from zero for values of ϵ sufficiently small. Hence equations (4:13) can be solved for w_5 and w_6 as unique power series in w_1, w_2, w_3 , and w_4 vanishing with these variables. Since by hypothesis y_1, y_2, y_3 , and y_4 take their initial values after a period, the same is true of w_1, w_2, w_3 , and w_4 . By the above argument this implies that w_5 and w_6 also take their initial values and the conclusion of the lemma follows immediately.

Another important lemma is the following:

LEMMA 4:2. A necessary and sufficient condition that the initial values of the dependent variables in equations (1:5) be real is that the initial values of the dependent variables in equations (4:6) be of the form

$$y_1 = a_1 + b_1 i, y_2 = a_1 - b_1 i, y_3 = a_3, y_4 = a_4$$

where a_1, b_1, a_3, a_4 are real.

The condition that the variables in equation (1:5) are

real implies that the variables in equation (2:2) are real by virtue of transformation (2:1). The condition is obviously sufficient in view of transformation (4:5). The condition is also necessary as may be seen by solving equations (4:5) for y_1 , y_2 , y_3 , and y_4 after giving arbitrary real values to the variables in the left members. The solutions have the form given in the lemma.

It will be shown further that the second equation of (4:10) follows from the three equations corresponding to $(j=1,3,4)$. If in the differential equations (4:6) y_1 is replaced by $\bar{y}_2$, y_2 by $\bar{y}_1$, y_3 by $\bar{y}_3$, y_4 by $\bar{y}_4$ and -1 by $\bar{1}$, the new system of differential equations is of the same form in the new variables and the $\bar{1}$ as equations (4:6) in the original variables and 1 . Furthermore by lemma (4:2) the initial values of $\bar{y}_1$, $\bar{y}_2$, $\bar{y}_3$, and $\bar{y}_4$ are the same as the initial values of y_1 , y_2 , y_3 , and y_4 with 1 replaced by $\bar{1}$. Therefore if the solution of equations (4:6) is

$$(4:14) \quad \begin{aligned} y_1 &= f_1(1, \tau), & y_2 &= f_2(1, \tau), & y_3 &= f_3(\tau), & y_4 &= f_4(\tau) \\ u_3 &= u_3(\tau), & v_3 &= v_3(\tau), \end{aligned}$$

then that of the transformed system is

$$(4:15) \quad \begin{aligned} \bar{y}_1 &= f_1(\bar{1}, \tau), & \bar{y}_2 &= f_2(\bar{1}, \tau), & \bar{y}_3 &= f_3(\tau), & \bar{y}_4 &= f_4(\tau) \\ u_3 &= u_3(\tau), & v_3 &= v_3(\tau). \end{aligned}$$

Now if the original notation is restored the first two equations of (4:15) become

$$(4:16) \quad y_2 = f_1(-1, \tau), \quad y_1 = f_2(-1, \tau).$$

Hence on comparing (4:16) and (4:14), it is evident that y_1 and y_2 are the same functions of τ except for the sign before the 1 . Hence if y_1 is a periodic function of τ , the same is true of y_2 .

In view of the above argument and the form of equations

(4:9), sufficient conditions that the solution of equations (4:6), and hence of (2:2), be periodic is, that the three equations

$$\begin{aligned}
 & \left(\frac{a}{2} + \nabla_1\right)(e^{2\pi i \delta} - 1) + \epsilon Q_1 = 0, \\
 (4:17) \quad & \nabla_3 \left(e^{\frac{2\pi \beta}{\alpha}(1+\delta)} - 1\right) + \epsilon Q_3 = 0, \\
 & \nabla_4 \left(e^{\frac{-2\pi \beta}{\alpha}(1+\delta)} - 1\right) + \epsilon Q_4 = 0,
 \end{aligned}$$

be satisfied for some value of each of the parameters $\nabla_1, \nabla_3, \nabla_4$, and δ . The Q_1, Q_3 , and Q_4 are power series in $\nabla_1, \nabla_3, \nabla_4$, and δ .

Equations (4:17) present a system of three equations in the five unknowns $\nabla_1, \nabla_3, \nabla_4, \delta$, and ϵ . It seems probable therefore that two of these quantities may be chosen arbitrarily. It will be shown first that it is impossible to choose δ arbitrarily and to solve the equations for ∇_1, ∇_3 , and ∇_4 as a power series in ϵ vanishing with ϵ . If $\nabla_1 = \epsilon = 0$ in the first equation the only value for δ which satisfies the equation is $\delta = 0$.

If however ∇_1 has any value consistent with lemma (4:2), then equations (4:17) can be solved for δ, ∇_3 , and ∇_4 as power series in ϵ which vanish with ϵ . Since the determinant of the coefficients of the linear terms in ∇_3 and ∇_4 in the last two equations of (4:17) is

$$\left(e^{\frac{2\pi \beta}{\alpha}} - 1\right)\left(e^{\frac{-2\pi \beta}{\alpha}} - 1\right) \neq 0,$$

these equations can be solved for ∇_3 and ∇_4 as unique power series in δ and ϵ . These values are then substituted in the first equation of (4:17) giving an equation in δ and ϵ which can be solved for δ as a convergent power series in ϵ vanishing with ϵ , since the coefficient of the linear term in δ is $2\pi i \left(\frac{a}{2} + \nabla_1\right)$ and there is no term independent of δ and ϵ . This

completes the proof of the first part of the following theorem:

THEOREM 4:1. There exist periodic solutions of equations (4:6) and hence of equations (1:5) for special initial values of y_3 , y_4 , and δ which are solutions of equations (4:17). The solution has the form

$$\delta = \epsilon h_1(\epsilon), \quad y_3 = \epsilon h_3(\epsilon), \quad y_4 = \epsilon h_4(\epsilon).$$

The solution of equations (1:5) is real provided the initial values of the variables in equation (4:6) satisfy the conditions of lemma (4:2).

The last statement in the theorem follows from the fact that solutions of differential equations of the type of (1:5) are real functions of their initial conditions and hence if the initial values of the variables are real the solution is real [2§31].

5. Existence of periodic solutions, case 2. Let the solutions of equation (2:6) be given by

$$(5:1) \quad \lambda_1 = \overline{\alpha} i, \quad \lambda_2 = -\overline{\alpha} i, \quad \lambda_3 = \overline{\beta} i, \quad \lambda_4 = -\overline{\beta} i,$$

where $\overline{\alpha}$ and $\overline{\beta}$ are positive real numbers and $i = (-1)^{1/2}$. In order that the general solution of equations (2:3) be periodic it is necessary and sufficient to impose the restriction that $\overline{\alpha}$ and $\overline{\beta}$ be commensurable. The question of the existence of commensurable roots will now be considered before proceeding with the main argument.

The condition that the roots (5:1) of equation (2:6) be commensurable is

$$(5:2) \quad \overline{\alpha}/\overline{\beta} = p/q,$$

where p and q are integers. It follows from equations (5:1) and (5:2) that

$$(5:3) \quad \frac{\overline{\alpha}^2}{\beta^2} = \frac{b - (b^2 - 4ac)^{1/2}}{b + (b^2 - 4ac)^{1/2}} = \frac{p^2}{q^2},$$

where a , b , and c are the coefficients which appear in equation (2:6), that equation being regarded as a quadratic in λ^2 .

We are interested primarily in developing sufficient conditions for the commensurability of $\overline{\alpha}$ and $\overline{\beta}$. With this aim in view it is desirable to find conditions on A , B , and C , the moments of inertia, such that $(b^2 - 4ac)^{1/2}$ will be a rational function of A , B , C , and n . By so doing it will be possible to find a further condition on n so that equation (5:3) holds for integral values of p and q .

The discriminant (2:7) is clearly equivalent to

$$(5:4) \quad b^2 - 4ac = [n^2(1 + m_1 m_2) + K_1 - K_2]^2 - 4n^2 K_1 + 4n^2 m_1 m_2 K_2 + 4n^2 (m_1 K_2 - m_2 K_1).$$

The right member will be a perfect square if the sum of the terms outside the brackets reduces to zero. This condition will be satisfied provided $m_1 = K_1/K_2$, which in view of the identities (1:6) implies the relation

$$(5:5) \quad C = 2B.$$

The condition (5:5) is possible only if $B < A$ since from dynamical considerations $A + B > C$.

If the value of $(b^2 - 4ac)^{1/2}$ from (5:4) and the value of b from (2:6) are substituted in equation (5:3), and identities (1:6) and equation (5:5) are taken into account, the resulting equation is,

$$(5:6) \quad p^2/q^2 = \left[-\frac{n^2}{K_2} \left(1 - \frac{2B}{A} \right) - \frac{B}{A} \right] / \left(\frac{n^2}{K_2} - 1 \right).$$

Choose n^2 to be a number satisfying the equation

$$(5:7) \quad \frac{n^2}{K_B} - 1 = \rho q^2 ,$$

where ρ is arbitrary and q is an integer. Substitute this value of n^2 in the numerator of equation (5:6) and then impose the condition that this numerator be equal to ρp^2 where p is an integer. The equation is

$$(5:8) \quad (1 + \rho q^2) \left(\frac{2B}{A} - 1 \right) - \frac{B}{A} = \rho p^2 .$$

The value of ρ is found from this equation to be

$$(5:9) \quad \rho = \left(1 - \frac{B}{A} \right) / \left[\frac{2B}{A} q^2 - (p^2 + q^2) \right] .$$

In order to be sure that ρ remains finite and has positive values for n large in accordance with equation (5:7) the following restrictions are imposed on A and p :

$$(5:10) \quad B < A < 2B , \quad p^2 < \left(\frac{2B}{A} - 1 \right) q^2 .$$

When the value of ρ from (5:9) is substituted in (5:8), (5:7) and finally in (5:6), it is evident that if (5:5) is satisfied then (5:2) is satisfied. This proves

THEOREM 5:1. There exist values of A , B , C , and n for which the roots of equation (2:6) are pure imaginaries and commensurable.

When the roots (5:1) are substituted in equations (2:5) the corresponding values for the constants are

$$(5:11) \quad \begin{aligned} \bar{c}_1^{(1)} &= \bar{h}_1 i \bar{c}_2^{(1)}, \quad \bar{c}_1^{(2)} = -\bar{h}_1 i \bar{c}_2^{(2)}, \quad \bar{c}_1^{(3)} = \bar{h}_2 i \bar{c}_2^{(3)}, \\ \bar{c}_1^{(4)} &= -\bar{h}_2 i \bar{c}_2^{(4)}, \end{aligned}$$

where

$$\bar{h}_1 = \frac{n(1+m_2)\bar{\alpha}}{-\bar{\alpha}^2 - m_2 n^2 - K_2}, \quad \text{or} \quad \bar{h}_1 = \frac{-\bar{\alpha}^2 + m_1 n^2 - K_1}{n(1-m_1)\bar{\alpha}},$$

$$\bar{h}_2 = \frac{n(1+m_2)\bar{\beta}}{-\bar{\beta}^2 - m_2 n^2 - K_2}, \quad \text{or} \quad \bar{h}_2 = \frac{-\bar{\beta}^2 + m_1 n^2 - K_1}{n(1-m_1)\bar{\beta}}.$$

The two sets of values for $\bar{h}_1$ and $\bar{h}_2$ are equivalent by virtue of the vanishing of the determinant in equations (2:5).

The general solution of equations (2:3) now has the form

$$\begin{aligned} v_1 &= \bar{h}_1 i \bar{c}_2^{(1)} e^{\bar{\alpha} i \tau} - \bar{h}_1 i \bar{c}_2^{(2)} e^{-\bar{\alpha} i \tau} + \bar{h}_2 i \bar{c}_2^{(3)} e^{\bar{\beta} i \tau} \\ &\quad - \bar{h}_2 i \bar{c}_2^{(4)} e^{-\bar{\beta} i \tau}, \\ v_2 &= \bar{c}_2^{(1)} e^{\bar{\alpha} i \tau} + \bar{c}_2^{(2)} e^{-\bar{\alpha} i \tau} + \bar{c}_2^{(3)} e^{\bar{\beta} i \tau} + \bar{c}_2^{(4)} e^{-\bar{\beta} i \tau}, \\ (5:12) \quad u_1 &= (n\bar{h}_1 + \bar{\alpha}) i \bar{c}_2^{(1)} e^{\bar{\alpha} i \tau} - (n\bar{h}_1 + \bar{\alpha}) i \bar{c}_2^{(2)} e^{-\bar{\alpha} i \tau} \\ &\quad + (n\bar{h}_2 + \bar{\beta}) i \bar{c}_2^{(3)} e^{\bar{\beta} i \tau} - (n\bar{h}_2 + \bar{\beta}) i \bar{c}_2^{(4)} e^{-\bar{\beta} i \tau}, \\ u_2 &= (n + \bar{\alpha}\bar{h}_1) \bar{c}_2^{(1)} e^{\bar{\alpha} i \tau} + (n + \bar{\alpha}\bar{h}_1) \bar{c}_2^{(2)} e^{-\bar{\alpha} i \tau} \\ &\quad + (n + \bar{\beta}\bar{h}_2) \bar{c}_2^{(3)} e^{\bar{\beta} i \tau} + (n + \bar{\beta}\bar{h}_2) \bar{c}_2^{(4)} e^{-\bar{\beta} i \tau}. \end{aligned}$$

The determinant of the coefficients of the c 's for $\tau = 0$

is

$$\Delta = -4(\bar{\beta}\bar{h}_2 - \bar{\alpha}\bar{h}_1)(\bar{\beta}\bar{h}_1 - \bar{\alpha}\bar{h}_2) \neq 0.$$

This determinant does not vanish, for each of its factors has the value

$$\bar{\beta}\bar{h}_2 - \bar{\alpha}\bar{h}_1 = \frac{\bar{\beta}^2 - \bar{\alpha}^2}{n(1-m)} \neq 0,$$

$$\bar{\beta}\bar{h}_1 - \bar{\alpha}\bar{h}_2 = \frac{n(1+m_2)\bar{\alpha}\bar{\beta}(\bar{\alpha}^2 - \bar{\beta}^2)}{(-\bar{\alpha}^2 - m_2 n^2 - K_2)(-\bar{\beta}^2 - m_2 n^2 - K_2)} \neq 0.$$

Neither of these factors is zero since by hypothesis the roots are

all unequal.

There is no loss of generality in supposing that t_0 is so chosen that $v_1 = 0$ when $\tau = 0$, since v_1 being related to γ_1 by equations (2:1) certainly vanishes at some time during the motion. Suppose the initial values of the v_1, v_2, u_1, u_2 , in (5:12) are 0, $\bar{b}_2, \bar{a}_1, \bar{a}_2$ respectively. If $\tau = 0$ in equations (5:12) and these equations are solved for $\bar{c}_2^{(j)}$ ($j = 1 \cdots 4$) the resulting solution is

$$\begin{aligned}
 \bar{c}_2^{(1)} &= \frac{\bar{a}_2 - \bar{b}_2(n + \bar{\beta}\bar{h}_2)}{2(\bar{\alpha}\bar{h}_1 - \bar{\beta}\bar{h}_2)} - \frac{\bar{a}_1\bar{h}_2 i}{2(\bar{\alpha}\bar{h}_1 - \bar{\beta}\bar{h}_2)} = \frac{A_1}{2} - \frac{B_{11}}{2}, \\
 \bar{c}_2^{(2)} &= \frac{\bar{a}_2 - \bar{b}_2(n + \bar{\beta}\bar{h}_2)}{2(\bar{\alpha}\bar{h}_1 - \bar{\beta}\bar{h}_2)} + \frac{\bar{a}_1\bar{h}_2 i}{2(\bar{\alpha}\bar{h}_2 - \bar{\beta}\bar{h}_1)} = \frac{A_1}{2} + \frac{B_{11}}{2}, \\
 \bar{c}_2^{(3)} &= \frac{-\bar{a}_2 + \bar{b}_2(n + \bar{\alpha}\bar{h}_1)}{2(\bar{\alpha}\bar{h}_1 - \bar{\beta}\bar{h}_2)} + \frac{\bar{a}_1\bar{h}_1 i}{2(\bar{\alpha}\bar{h}_2 - \bar{\beta}\bar{h}_1)} = \frac{A_2}{2} + \frac{B_{21}}{2}, \\
 \bar{c}_2^{(4)} &= \frac{-\bar{a}_2 + \bar{b}_2(n + \bar{\alpha}\bar{h}_1)}{2(\bar{\alpha}\bar{h}_1 - \bar{\beta}\bar{h}_2)} - \frac{\bar{a}_1\bar{h}_1 i}{2(\bar{\alpha}\bar{h}_2 - \bar{\beta}\bar{h}_1)} = \frac{A_2}{2} - \frac{B_{21}}{2}.
 \end{aligned}
 \tag{5:13}$$

When the values for the $\bar{c}_2^{(j)}$ are substituted in equations (5:12), these equations could be written in terms of the trigonometric functions with real coefficients. It will not be necessary however to write out this solution explicitly.

In order to prove the existence of periodic solutions for the case under consideration in this section it is convenient to again make a transformation analogous to (4:5). The transformation for the present case is

$$\begin{aligned}
 v_1 &= \bar{h}_1 i(y_1 - y_2) + \bar{h}_2 i(y_3 - y_4), \\
 v_2 &= y_1 + y_2 + y_3 + y_4, \\
 u_1 &= (n\bar{h}_1 + \bar{\alpha})i(y_1 - y_2) + (n\bar{h}_2 + \bar{\beta})i(y_3 - y_4), \\
 u_2 &= (n + \bar{\alpha}\bar{h}_1)(y_1 + y_2) + (n + \bar{\beta}\bar{h}_2)(y_3 + y_4).
 \end{aligned}
 \tag{5:14}$$

If transformation (5:14) is substituted into equations (2:2) and the same steps taken as in the derivation of equations (4:6) the resulting equations are

$$\begin{aligned}
 \frac{dy_1}{d\tau} &= (1 + \delta) \overline{\alpha} i y_1 + (1 + \delta) \frac{(\overline{R}_2 - \overline{R}_1)}{2} i \epsilon, \\
 \frac{dy_2}{d\tau} &= - (1 + \delta) \overline{\alpha} i y_2 - (1 + \delta) \frac{(\overline{R}_2 + \overline{R}_1)}{2} i \epsilon, \\
 \frac{dy_3}{d\tau} &= (1 + \delta) \overline{\beta} i y_3 + (1 + \delta) \frac{(\overline{R}_4 - \overline{R}_3)}{2} i \epsilon, \\
 \frac{dy_4}{d\tau} &= - (1 + \delta) \overline{\beta} i y_4 - (1 + \delta) \frac{(\overline{R}_4 + \overline{R}_3)}{2} i \epsilon, \\
 \frac{du_3}{d\tau} &= - m_3 (1 + \delta) u_1 u_2 \epsilon, \\
 \frac{dv_3}{d\tau} &= (1 + \delta) (v_1 u_2 - u_2 v_1),
 \end{aligned}
 \tag{5:15}$$

where

$$\begin{aligned}
 \overline{R}_1 &= (\overline{p}_1 u_3 + \overline{q}_1 v_3)(y_1 - y_2) + (\overline{r}_1 u_3 + \overline{s}_1 v_3)(y_3 - y_4), \\
 \overline{R}_2 &= (\overline{p}_2 u_3 - \overline{q}_2 v_3)(y_1 + y_2) + (\overline{r}_2 u_3 - \overline{s}_2 v_3)(y_3 + y_4), \\
 \overline{R}_3 &= (\overline{p}_3 u_3 + \overline{q}_3 v_3)(y_1 - y_2) + (\overline{r}_3 u_3 + \overline{s}_3 v_3)(y_3 - y_4), \\
 \overline{R}_4 &= (\overline{p}_4 u_3 - \overline{q}_4 v_3)(y_1 + y_2) + (\overline{r}_4 u_3 - \overline{s}_4 v_3)(y_3 + y_4), \\
 \overline{p}_1 &= \frac{m_2(n\overline{h}_1 + \overline{\alpha}) - (n + \overline{\beta}\overline{h}_2)\overline{h}_1}{\overline{\alpha}\overline{h}_1 - \overline{\beta}\overline{h}_2}, \quad \overline{q}_1 = \frac{(n + \overline{\beta}\overline{h}_2)(n\overline{h}_1 + \overline{\alpha})}{\overline{\alpha}\overline{h}_1 - \overline{\beta}\overline{h}_2}, \\
 \overline{r}_1 &= \frac{m_2(n\overline{h}_2 + \overline{\beta}) - (n + \overline{\beta}\overline{h}_2)\overline{h}_2}{\overline{\alpha}\overline{h}_1 - \overline{\beta}\overline{h}_2}, \quad \overline{s}_1 = \frac{(n + \overline{\beta}\overline{h}_2)(n\overline{h}_2 + \overline{\beta})}{\overline{\alpha}\overline{h}_1 - \overline{\beta}\overline{h}_2}, \\
 \overline{r}_2 &= \frac{\overline{h}_2 m_1(n + \overline{\alpha}\overline{h}_1) + n\overline{h}_2 + \overline{\beta}}{\overline{\alpha}\overline{h}_2 - \overline{\beta}\overline{h}_1}, \quad \overline{q}_2 = \frac{(n\overline{h}_2 + \overline{\beta})(n + \overline{\alpha}\overline{h}_1)}{\overline{\alpha}\overline{h}_2 - \overline{\beta}\overline{h}_1}, \\
 \overline{r}_2 &= \frac{\overline{h}_2 m_1(n + \overline{\beta}\overline{h}_2) + n\overline{h}_2 + \overline{\beta}}{\overline{\alpha}\overline{h}_2 - \overline{\beta}\overline{h}_1}, \quad \overline{s}_2 = \frac{(n\overline{h}_2 + \overline{\beta})(n + \overline{\beta}\overline{h}_2)}{\overline{\alpha}\overline{h}_2 - \overline{\beta}\overline{h}_1}, \\
 \overline{p}_3 &= \frac{m_2(n\overline{h}_1 + \overline{\alpha}) - (n + \overline{\alpha}\overline{h}_1)\overline{h}_1}{\overline{\beta}\overline{h}_2 - \overline{\alpha}\overline{h}_1}, \quad \overline{q}_3 = \frac{(n + \overline{\alpha}\overline{h}_1)(n\overline{h}_1 + \overline{\alpha})}{\overline{\beta}\overline{h}_2 - \overline{\alpha}\overline{h}_1}, \\
 \overline{r}_3 &= \frac{m_2(n\overline{h}_2 + \overline{\beta}) - (n + \overline{\alpha}\overline{h}_1)\overline{h}_2}{\overline{\beta}\overline{h}_2 - \overline{\alpha}\overline{h}_1}, \quad \overline{s}_3 = \frac{(n + \overline{\alpha}\overline{h}_1)(n\overline{h}_2 + \overline{\beta})}{\overline{\beta}\overline{h}_2 - \overline{\alpha}\overline{h}_1}, \\
 \overline{p}_4 &= \frac{\overline{h}_1 m_1(n + \overline{\alpha}\overline{h}_1) + n\overline{h}_1 + \overline{\alpha}}{\overline{\beta}\overline{h}_1 - \overline{\alpha}\overline{h}_2}, \quad \overline{q}_4 = \frac{(n\overline{h}_1 + \overline{\alpha})(n + \overline{\alpha}\overline{h}_1)}{\overline{\beta}\overline{h}_1 - \overline{\alpha}\overline{h}_2},
 \end{aligned}$$

$$\bar{r}_4 = \frac{\bar{h}_1 m_1 (n + \beta \bar{h}_2) + n \bar{h}_1 + \bar{\alpha}}{\beta \bar{h}_1 - \bar{\alpha} \bar{h}_2}, \quad \bar{s}_4 = \frac{(n \bar{h}_1 + \bar{\alpha})(n + \beta \bar{h}_2)}{\beta \bar{h}_1 - \bar{\alpha} \bar{h}_2}.$$

Following the same procedure as in section 4, the solution of equations (5:15) is of the form

$$\begin{aligned} (5:16) \quad y_1 &= (\bar{c}_2^{(1)} + \nabla_1) e^{\bar{\alpha} i(1+\delta)T} + \epsilon P_1(\nabla_1, \nabla_2, \nabla_3, \nabla_4, \rho, \gamma, \delta, \epsilon; T), \\ y_2 &= (\bar{c}_2^{(2)} + \nabla_2) e^{-\bar{\alpha} i(1+\delta)T} + \epsilon P_2(\nabla_1, \nabla_2, \nabla_3, \nabla_4, \rho, \gamma, \delta, \epsilon; T), \\ y_3 &= (\bar{c}_2^{(3)} + \nabla_3) e^{\bar{\beta} i(1+\delta)T} + \epsilon P_3(\nabla_1, \nabla_2, \nabla_3, \nabla_4, \rho, \gamma, \delta, \epsilon; T), \\ y_4 &= (\bar{c}_2^{(4)} + \nabla_4) e^{-\bar{\beta} i(1+\delta)T} + \epsilon P_4(\nabla_1, \nabla_2, \nabla_3, \nabla_4, \rho, \gamma, \delta, \epsilon; T), \\ u_3 &= b + \rho + \epsilon P_5(\nabla_1, \nabla_2, \nabla_3, \nabla_4, \rho, \gamma, \delta, \epsilon; T), \\ v_3 &= c + \gamma + \epsilon P_6(\nabla_1, \nabla_2, \nabla_3, \nabla_4, \rho, \gamma, \delta, \epsilon; T), \end{aligned}$$

where the P_j are power series in the arguments exhibited.

By virtue of equations (4:11) it follows, as in the preceding case, that u_3 and v_3 will be periodic if y_1, y_2, y_3 , and y_4 satisfy the condition for periodicity. There will be no restrictions on the ρ and γ in the work that is to follow and henceforth they may be regarded as absorbed in the b and c respectively. It should be remembered however that the initial value of v_3 is negative by virtue of (2:1) and the fact that $|\gamma_3| \leq 1$.

A lemma similar to lemma (4:2) follows:

LEMMA 5:1. A necessary and sufficient condition that the initial values of the dependent variables in equations (1:5) be real, is that the initial values of the dependent variables in equations (5:15) be of the form

$$y_1 = a_1 + b_1 i, \quad y_2 = a_1 - b_1 i, \quad y_3 = a_3 + b_3 i, \quad y_4 = a_3 - b_3 i,$$

where a_1, b_1, a_3, b_3 are real.

The condition is certainly sufficient in view of transformation (5:14). The condition is also necessary since the solution for the initial values of the y 's is given by (5:13) for the most general real initial values of the variables v_1, v_2, u_1 , and u_2 . They have precisely the form given in the lemma.

With the aid of the above lemma it can be shown further, that the y_2 and y_4 of equations (5:16) are periodic functions with period $2\pi p/\alpha = 2\pi q/\rho$ if the same is true for y_1 and y_3 . If in the differential equations (5:15) y_1 is replaced by $\bar{y}_2$, y_2 by $\bar{y}_1$, y_3 by $\bar{y}_4$, y_4 by $\bar{y}_3$, and $-i$ by $\bar{i}$, the resulting system of differential equations is the same as (5:15) except that the symbol $\bar{i}$ replaces the symbol i throughout. By lemma (5:1) the initial values of the $\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{y}_4$ are the same respectively as the initial values of y_1, y_2, y_3 , and y_4 with i replaced by $\bar{i}$. Hence if the solution of equations (5:15) is of the form

$$(5:17) \quad \begin{aligned} y_1 &= f_1(i, \tau), \quad y_2 = f_2(i, \tau), \quad y_3 = f_3(i, \tau), \quad y_4 = f_4(i, \tau), \\ u_3 &= u_3(\tau), \quad v_3 = v_3(\tau), \end{aligned}$$

then the solution of the new system is of the form

$$(5:18) \quad \bar{y}_1 = f_1(\bar{i}, \tau), \quad \bar{y}_2 = f_2(\bar{i}, \tau), \quad \bar{y}_3 = f_3(\bar{i}, \tau), \quad \bar{y}_4 = f_4(\bar{i}, \tau).$$

Restoring the original notation in (5:18) we have

$$(5:19) \quad y_2 = f_1(-i, \tau), \quad y_1 = f_2(-i, \tau), \quad y_4 = f_3(-i, \tau), \quad y_3 = f_4(-i, \tau).$$

Now on comparing equations (5:17) and (5:19) it is seen that y_2 and y_4 are the same functions of τ respectively as y_1 and y_3 except for the sign before the i . It follows that y_2 and y_4 are periodic if y_1 and y_3 are.

The above argument shows that sufficient conditions for

the periodicity of the solution (5:16) are the following two equations:

$$(5:20) \quad \begin{aligned} (\bar{c}_2^{(1)} + \nabla_1)(e^{2\pi i(1+\delta)} - 1) + \epsilon Q_1(\nabla_1, \nabla_3, \delta, \epsilon) &= 0, \\ (\bar{c}_2^{(3)} + \nabla_3)(e^{2\pi i(1+\delta)} - 1) + \epsilon Q_3(\nabla_1, \nabla_3, \delta, \epsilon) &= 0, \end{aligned}$$

which are derived by imposing the condition of periodicity on the first and third equations of (5:16).

The first equation of (5:20) has a term of the first degree in δ viz. $2\pi i \delta \bar{c}_2^{(1)}$. Hence this equation can be solved for δ as a power series in ∇_3 and ϵ having ϵ as a factor and a term in ϵ alone. The latter statement is justified by noticing that Q_1 has a term independent of ∇_3 , δ , and ϵ which is due to the resonance [2875] occurring when the term independent of ϵ in (5:16) is substituted in (5:15) for the purpose of computing higher powers. This term gives rise to a term in ϵ alone in the δ series. Now let the value of δ obtained from the first equation of (5:20) be substituted in the second equation. The resulting equation, after dividing by the factor ϵ , has a term independent of ∇_3 and ϵ if $\bar{c}_2^{(3)} \neq 0$. Hence the equation does not have a solution for ∇_3 as a power series in ϵ vanishing with ϵ .

If $\bar{c}_2^{(3)} = 0$ then Q_3 has no term independent of ϵ and ∇_3 for there is no resonance occurring in the first degree term in ϵ in the computation of y_3 from (5:15). These terms being periodic they cancel in the Q_3 , leaving no term independent of ϵ or ∇_3 . Therefore, after the factor ϵ has been removed from the second equation of (5:20) there exists a solution of the resulting equation for ∇_3 as a power series in ϵ vanishing with ϵ . When this series for ∇_3 is substituted in the δ series the solution of equations (5:20) has the form

$$(5:21) \quad \delta = \epsilon h_1(\epsilon), \quad \nabla_3 = \epsilon h_2(\epsilon).$$

By virtue of equations (5:13) however, the condition $\bar{c}_2^{(3)} = 0$ implies also that $\bar{c}_2^{(4)} = 0$ and $\bar{c}_2^{(2)} = \bar{c}_2^{(1)} = \bar{b}_2/2$. But these are precisely the initial values imposed upon (4:3) to obtain the solution (4:4). Hence (5:12) is of the same form as (4:4) and does not contain terms involving the roots $(\beta 1)$ and $(-\beta 1)$ of the characteristic equation. It will be seen in the next chapter that these terms will also be absent in the higher degree terms of the solution. The results of this section may be summed in the following theorem:

THEOREM 5:1. There exist periodic solutions of equations (5:15), and hence of equations (1:5) for the case when the roots of the characteristic equation (2:6) are all pure imaginary and commensurable. The solution (5:12) for $\epsilon = \delta = 0$ has the period $2\pi/\alpha$ and is of the same form as the corresponding solution (4:4) for the case when two of the characteristic roots are real. The solution of equations (1:5) is real provided the initial values of the variables in equations (5:15) satisfy the conditions of lemma (5:1).

The last statement of the theorem follows in the same manner as the corresponding statement in theorem 4:1.

The point of view in this and the preceding section has been to classify the solutions according to the nature of the roots of the characteristic equation. It is clear then that there is no essential difference in the development of sections 4 and 5 for a rigid body spinning with its center of gravity below the point of support, since it was proved in section 3 that the characteristic equation may have four pure imaginary roots or two real and two pure imaginary in either case. The same remark holds for

the following sections. It will be assumed in the work which is to follow that the center of gravity is above the point of support.

CHAPTER II

DIRECT CONSTRUCTION OF THE SOLUTIONS

6. Case 1. In the direct construction of the solution of equations (1:5) for the case in which two of the characteristic roots are real, and two are pure imaginaries, it is convenient, first to find the solution of equations (4:6), then by means of transformation (4:5) to find the solution of equations (2:2), and finally by means of transformation (2:1) to find the solution of equations (1:5).

The solution of equations (4:6) as a power series in ϵ is of the form

$$(6:1) \quad y_1 = y_1^{(j)} \epsilon^j, \quad \begin{matrix} i = (1,2,3,4) \\ j = (0,1,\dots) \end{matrix}, \quad u_3 = u_3^{(j)} \epsilon^j, \quad v_3 = v_3^{(j)} \epsilon^j, \\ \delta = \delta_j \epsilon^j,$$

where the repeated index denotes a sum according to the tensor analysis notation. The corresponding solution of equations (2:2) after making the transformation (4:5) is of the form

$$(6:2) \quad u_1 = u_1^{(j)} \epsilon^j, \quad \begin{matrix} i = (1,2,3) \\ j = (0,1,2,\dots) \end{matrix}, \quad v_1 = v_1^{(j)} \epsilon^j, \quad \delta = \delta_j \epsilon^j.$$

The terms in the solution (6:2) independent of ϵ , were found in section 4 to be:

$$(6:3) \quad \begin{aligned} u_1^{(0)} &= - (nh_1 + \alpha)a \sin \alpha \tau, & v_1^{(0)} &= - ah_1 \sin \alpha \tau, \\ u_2^{(0)} &= (n + \alpha h_1)a \cos \alpha \tau, & v_2^{(0)} &= a \cos \alpha \tau, \\ u_3^{(0)} &= b, & v_3^{(0)} &= c. \end{aligned}$$

The corresponding terms of solution (6:1) which are independent of ϵ were found to be

$$(6:4) \quad \begin{aligned} y_1^{(0)} &= \frac{a}{2} e^{-\alpha i \tau}, & y_3^{(0)} &= 0, \\ y_2^{(0)} &= \frac{a}{2} e^{-\alpha i \tau}, & y_4^{(0)} &= 0, \\ u_3^{(0)} &= b, & v_3^{(0)} &= c. \end{aligned}$$

The coefficients of ϵ in the solution (6:1) are found by substituting these equations in the differential equations (4:6) and equating the coefficients of ϵ in the two members of each of the resulting equations. The resulting differential equations are

$$(6:5) \quad \begin{aligned} \frac{dy_1^{(1)}}{d\tau} &= \alpha i y_1^{(1)} + \left[\delta_1 \alpha + \frac{p_2 b - q_2 c}{2} - \frac{(p_1 b + q_1 c)}{2} \right] \frac{1}{2} e^{\alpha i \tau} \\ &\quad + \left[\frac{p_2 b - q_2 c}{2} + \frac{p_1 b + q_1 c}{2} \right] \frac{1}{2} e^{-\alpha i \tau}, \\ \frac{dy_2^{(1)}}{d\tau} &= -\alpha i y_2^{(1)} - \left[\delta_1 \alpha + \frac{p_2 b - q_2 c}{2} - \frac{(p_1 b + q_1 c)}{2} \right] \frac{1}{2} e^{-\alpha i \tau} \\ &\quad - \left[\frac{p_2 b - q_2 c}{2} + \frac{p_1 b + q_1 c}{2} \right] \frac{1}{2} e^{\alpha i \tau}, \\ \frac{dy_3^{(1)}}{d\tau} &= \beta y_3^{(1)} - \frac{(p_3 b + q_3 c)}{2} a \sin \alpha \tau + \frac{(p_4 b - q_4 c)}{2} a \cos \alpha \tau, \\ \frac{dy_4^{(1)}}{d\tau} &= -\beta y_4^{(1)} - \frac{(p_3 b + q_3 c)}{2} a \sin \alpha \tau - \frac{(p_4 b - q_4 c)}{2} a \cos \alpha \tau, \\ \frac{du_3^{(1)}}{d\tau} &= \frac{m_3}{2} (n h_1 + \alpha) (n + \alpha h_1) a^2 \sin 2\alpha \tau, \\ \frac{dv_3^{(1)}}{d\tau} &= \frac{a^2 \alpha (1 - h_1^2)}{2} \sin 2\alpha \tau. \end{aligned}$$

In order that the solution of the first two equations of (6:5) shall be periodic, the coefficient of $e^{\alpha i \tau}$ in the first

equation and the coefficient of $e^{-\alpha i \tau}$ in the second equation must vanish. Both of these conditions will be satisfied if δ_1 has the value

$$(6:6) \quad \delta_1 = \frac{(p_1 - p_2)b + (q_1 + q_2)c}{2\alpha}.$$

If condition (6:6) is satisfied the solution of differential equations (6:5) is of the form:

$$(6:7) \quad \begin{aligned} y_1^{(1)} &= K_1^{(1)} e^{\alpha i \tau} + a_1^{(1)} e^{-\alpha i \tau}, \\ y_2^{(1)} &= K_2^{(1)} e^{-\alpha i \tau} + b_2^{(1)} e^{\alpha i \tau}, \\ y_3^{(1)} &= a_3^{(1)} \cos \alpha \tau + b_3^{(1)} \sin \alpha \tau, \\ y_4^{(1)} &= a_4^{(1)} \cos \alpha \tau + b_4^{(1)} \sin \alpha \tau, \\ u_3^{(1)} &= \frac{-m_3 a^2 (n h_1 + \alpha)(n + \alpha h_1)(\cos 2\alpha \tau)}{4\alpha}, \\ v_3^{(1)} &= \frac{a^2 (h_1^2 - 1)(\cos 2\alpha \tau)}{4}, \end{aligned}$$

where $K_1^{(1)}$ and $K_2^{(1)}$ are arbitrary constants and the other constants are determined by substituting (6:7) into (6:5) and determining the coefficients such that the resulting equation will be an identity in τ . Their values are

$$(6:8) \quad \begin{aligned} a_1^{(1)} &= -\frac{a}{8\alpha} [(p_2 + p_1)b + (q_1 - q_2)c] = b_2^{(1)}, \\ b_3^{(1)} &= \frac{(p_3\beta + p_4\alpha)ab + (q_3\beta - q_4\alpha)ac}{2(\alpha^2 + \beta^2)} = -b_4^{(1)}, \\ a_3^{(1)} &= \frac{(p_3\alpha - p_4\beta)ab + (q_3\alpha + q_4\beta)ac}{2(\alpha^2 + \beta^2)} = a_4^{(1)}. \end{aligned}$$

The arbitrary constants in the general solution for the third and fourth equations of (6:5) are taken equal to zero since they multiply $e^{\beta \tau}$ and $e^{-\beta \tau}$ respectively, and these terms are not periodic. The constants $K_1^{(1)}$ and $K_2^{(1)}$ must be conjugate

imaginaries by lemma 4:2. The further condition that $v_1^{(1)} = v_2^{(1)} = 0$ when $\tau = 0$ is sufficient to determine $K_1^{(1)}$ and $K_2^{(1)}$. Their values are

$$K_1^{(1)} = K_2^{(1)} = - (a_1^{(1)} + a_3^{(1)}) .$$

The corresponding first degree coefficients of the solution (6:2) are:

$$\begin{aligned} u_1^{(1)} &= A_{11}^{(1)} \sin \alpha \tau, & v_1^{(1)} &= B_{11}^{(1)} \sin \alpha \tau, \\ (6:9) \quad u_2^{(1)} &= A_{21}^{(1)} \cos \alpha \tau, & v_2^{(1)} &= 0, \\ u_3^{(1)} &= A_{32}^{(1)} \cos 2 \alpha \tau, & v_3^{(1)} &= B_{32}^{(1)} \cos 2 \alpha \tau, \end{aligned}$$

where

$$\begin{aligned} A_{11}^{(1)} &= 2 \left[(nh_1 + \alpha)(2a_1^{(1)} + a_3^{(1)}) + (nh_2 + \beta)b_3^{(1)} \right], \\ A_{21} &= -2(\alpha h_1 + \beta h_2)a_3^{(1)}, & A_{32} &= \frac{-m_3 a^2 (nh_1 + \alpha)(n + \alpha h_1)}{4\alpha}, \\ B_{11}^{(1)} &= 2h_1(2a_1^{(1)} + a_3^{(1)}) + 2h_2 b_3^{(1)}, & B_{32}^{(1)} &= \frac{a^2(h_1^2 - 1)}{4}. \end{aligned}$$

In order to find the form of the general term in the solution it will be necessary to find one more term of the solution (6:1). The differential equations for the second degree terms are:

$$\begin{aligned} \frac{dy_1^{(2)}}{d\tau} &= \alpha i y_1^{(2)} + \left\{ \frac{\delta_2 \alpha a}{2} - \delta_1 \alpha (a_1^{(1)} + a_3^{(1)}) + M_1 - N_1 \right. \\ &\quad \left. + \frac{a}{8} (p_2 A_{32}^{(1)} - q_2 B_{32}^{(1)} + p_1 A_{32}^{(1)} + q_1 B_{32}^{(1)}) \right\} i e^{\alpha i \tau} \\ &\quad + J_1 i e^{-\alpha i \tau} + J_2 i e^{3\alpha i \tau} + J_3 i e^{-3\alpha i \tau}, \\ (6:10') \quad \frac{dy_2^{(2)}}{d\tau} &= -\alpha i y_2^{(2)} - \left\{ \frac{\delta_2 \alpha a}{2} - \delta_1 \alpha (a_1^{(1)} + a_3^{(1)}) + M_1 - N_1 \right\} i e^{-\alpha i \tau} \\ &\quad - \left\{ \frac{a}{8} (p_2 A_{32}^{(1)} - q_2 B_{32}^{(1)} + p_1 A_{32}^{(1)} + q_1 B_{32}^{(1)}) \right\} \\ &\quad - J_1 i e^{\alpha i \tau} + J_3 i e^{3\alpha i \tau} + J_2 i e^{-3\alpha i \tau}, \end{aligned}$$

$$\begin{aligned}
 \frac{dy_3^{(2)}}{d\tau} &= \beta y_3^{(2)} + M_3 \cos \alpha \tau + N_3 \sin \alpha \tau - J_4 \sin 3\alpha \tau + J_5 \cos 3\alpha \tau, \\
 \frac{dy_4^{(2)}}{d\tau} &= -\beta y_4^{(2)} - M_3 \cos \alpha \tau + N_3 \sin \alpha \tau - J_4 \sin 3\alpha \tau - J_5 \cos 3\alpha \tau, \\
 (6:10'') \quad \frac{du_3^{(2)}}{d\tau} &= M_5 \sin 2\alpha \tau, \\
 \frac{dv_3^{(2)}}{d\tau} &= M_6 \sin 2\alpha \tau,
 \end{aligned}$$

where the constants are defined as follows:

$$\begin{aligned}
 M_1 &= \frac{1}{2}[(q_2 c - p_2 b + r_2 b - s_2 c)a_3^{(1)} + \frac{\delta_1}{2}(p_2 b - q_2 c)a], \\
 N_1 &= \frac{1}{2}[-(p_1 b + q_1 c)(2a_1^{(1)} + a_3^{(1)}) - (r_1 b + s_1 c)b_3^{(1)} \\
 &\quad + \frac{\delta_1}{2}(p_1 b + q_1 c)a], \\
 M_3 &= (\delta_1 \beta - p_4 b + q_4 c + r_4 b - s_4 c)a_3^{(1)} + J_5 + \frac{\delta_1 a}{2}(p_4 b - q_4 c), \\
 N_3 &= (\delta_1 \beta + r_3 b + s_3 c)b_3^{(1)} + (p_3 b + q_3 c)(2a_1^{(1)} + a_3^{(1)}) - \frac{\delta_1 a}{2}(p_3 b + q_3 c) + J_4, \\
 M_5 &= -m_3(nh_1 + \alpha)a(\alpha h_1 + \beta h_2 + n + \alpha h_1)a_3^{(1)} - 2m_3(nh_1 + \alpha)(n + \alpha h_1)aa_1^{(1)} \\
 &\quad - m_3(nh_2 + \beta)(n + \alpha h_1)ab_3^{(1)} + \frac{m_3}{2}(n + \alpha h_1)(nh_1 + \alpha)a^2\delta_1, \\
 M_6 &= (2a\alpha h_1^2 + a\beta h_1 h_2 - a\alpha)a_3^{(1)} + (2a\alpha h_1^2 - 2a\alpha)a_1^{(1)} + (a\alpha h_1 h_2 - a\beta)b_3^{(1)} \\
 &\quad + \frac{\delta_1 \alpha a^2}{2}(1 - h_1^2), \\
 J_1 &= \delta_1 \alpha a_1^{(1)} + M_1 + N_1 + J_2, \quad J_2 = \frac{a}{8}[p_2 - p_1]A_{32}^{(1)} - (q_1 + q_2)B_{32}^{(1)}, \\
 J_3 &= \frac{a}{8}[p_2 + p_1]A_{32}^{(1)} + (q_1 - q_2)B_{32}^{(1)}, \quad J_4 = \frac{a}{4}(p_3 A_{32}^{(1)} + q_3 B_{32}^{(1)}), \\
 J_5 &= \frac{a}{4}(p_4 A_{32}^{(1)} - q_4 B_{32}^{(1)}).
 \end{aligned}$$

A necessary and sufficient condition that the solution of equations (6:10) be periodic is that the coefficient of $e^{\alpha i \tau}$ in the first equation and of $e^{-\alpha i \tau}$ in the second equation be zero. From the definition of J_3 and the form of the first two equations

of (6:10') it is evident that both of these conditions will be satisfied if δ_2 has the value

$$(6:11) \quad \delta_2 = + \frac{2}{a\alpha} [\delta_1 \alpha (a_1^{(1)} + a_3^{(1)}) - M_1 + N_1 - J_3] .$$

If the condition (6:11) is satisfied the solution of equations (6:10') and (6:10'') has the form:

$$(6:12) \quad \begin{aligned} y_1(2) &= K_1^{(2)} e^{\alpha 1\tau} + a_1^{(2)} e^{-\alpha 1\tau} + c_1^{(2)} e^{3\alpha 1\tau} + d_1^{(2)} e^{-3\alpha 1\tau}, \\ y_2(2) &= K_2^{(2)} e^{-\alpha 1\tau} + b_2^{(2)} e^{\alpha 1\tau} + c_2^{(2)} e^{3\alpha 1\tau} + d_2^{(2)} e^{-3\alpha 1\tau}, \\ y_3(2) &= a_3^{(2)} \cos \alpha \tau + b_3^{(2)} \sin \alpha \tau + c_3^{(2)} \cos 3\alpha \tau + d_3^{(2)} \sin 3\alpha \tau, \\ y_4(2) &= a_4^{(2)} \cos \alpha \tau + b_4^{(2)} \sin \alpha \tau + c_4^{(2)} \cos 3\alpha \tau + d_4^{(2)} \sin 3\alpha \tau, \\ u_3(2) &= - \frac{M_5}{2\alpha} (\cos 2\alpha \tau) , \\ v_3(2) &= - \frac{M_6}{2\alpha} (\cos 2\alpha \tau) , \end{aligned}$$

where $K_1^{(2)}$ and $K_2^{(2)}$ are arbitrary constants. The arbitrary constants in the third and fourth equations of (6:12) must vanish since they multiply non-periodic terms. The coefficients in (6:12) are determined by substituting these equations in (6:10) and requiring that the resulting equations be identities in τ . Their values are:

$$\begin{aligned} a_1(2) &= \frac{-J_1}{2\alpha} = b_2(2), \quad c_1(2) = \frac{J_2}{2\alpha} = d_2(2), \quad d_1(2) = \frac{-J_3}{4\alpha} = c_2(2), \\ a_3(2) &= - \frac{(\alpha M_3 + \alpha N_3)}{\alpha^2 + \beta^2} = a_4(2), \quad b_3(2) = \frac{\alpha M_3 - \beta N_3}{\alpha^2 + \beta^2} = -b_4(2), \\ c_3(2) &= \frac{3\alpha J_4 - \beta J_5}{9\alpha^2 + \beta^2} = c_4(2), \quad d_3(2) = \frac{3\alpha J_5 + \beta J_4}{9\alpha^2 + \beta^2} = -d_4(2). \end{aligned}$$

The constants $K_1^{(2)}$ and $K_2^{(2)}$ are determined as in the case of the first degree terms so that $v_1(2) = v_2(2) = 0$ when $\tau = 0$. Their values are

$$(6:13) \quad K_1^{(2)} = K_2^{(2)} = - (a_1^{(2)} + a_3^{(2)} + c_1^{(2)} + d_1^{(2)} + c_3^{(2)}).$$

The second degree coefficients of the solution (6:2) may now be written and are of the form

$$\begin{aligned}
 u_1^{(2)} &= A_{11}^{(2)} \sin \alpha \tau + A_{13}^{(2)} \sin 3\alpha \tau, \\
 u_2^{(2)} &= A_{21}^{(2)} \cos \alpha \tau + A_{23}^{(2)} \cos 3\alpha \tau, \\
 u_3^{(2)} &= A_{32}^{(2)} \cos 2\alpha \tau, \\
 (6:14) \quad v_1^{(2)} &= B_{11}^{(2)} \sin \alpha \tau + B_{13}^{(2)} \sin 3\alpha \tau, \\
 v_2^{(2)} &= B_{21}^{(2)} (\cos \alpha \tau - \cos 3\alpha \tau), \\
 v_3^{(2)} &= B_{32}^{(2)} \cos 2\alpha \tau,
 \end{aligned}$$

where the coefficients have the values

$$\begin{aligned}
 A_{11}^{(2)} &= 2[(nh_1 + \alpha)(2a_1^{(2)} + a_3^{(2)} + c_1^{(2)} + d_1^{(2)} + c_3^{(2)}) \\
 &\quad + (nh_2 + \beta)b_3^{(2)}], \\
 A_{13}^{(2)} &= 2[(nh_1 + \alpha)(d_1^{(2)} - c_1^{(2)}) + (nh_2 + \beta)d_3^{(2)}], \\
 A_{21}^{(2)} &= -2[(\alpha h_1 + \beta h_2)a_3^{(2)} + (n + \alpha h_1)(c_1^{(2)} + d_1^{(2)} + c_3^{(2)})], \\
 A_{23}^{(2)} &= 2[(n + \alpha h_1)(c_1^{(2)} + d_1^{(2)}) + (n - \beta h_2)c_3^{(2)}], \\
 A_{32}^{(2)} &= -\frac{M_5}{2\alpha}, \quad B_{11}^{(2)} = 2h_1(2a_1^{(2)} + a_3^{(2)} + c_1^{(2)} + d_1^{(2)} + c_3^{(2)}) + 2h_2b_3^{(2)}, \\
 B_{13}^{(2)} &= 2(h_1d_1^{(2)} + h_2d_3^{(2)} - h_1c_1^{(2)}), \\
 B_{21}^{(2)} &= -2(c_1^{(2)} + d_1^{(2)} + c_3^{(2)}), \quad B_{32}^{(2)} = -\frac{M_6}{2\alpha}.
 \end{aligned}$$

To find the general term of the solution, it will be assumed that the coefficients $y_1^{(j)} \dots y_4^{(j)} u_3^{(j)} u_4^{(j)}$ $j=[1, \dots, (n-1)]$ for equations (6:1) have been computed and have the following properties:

1. The $y_1^{(j)}$ and $y_2^{(j)}$ are sums of exponentials of the

form $e^{(2m+1)\alpha\tau}$, $e^{-(2m+1)\alpha\tau}$ ($m=0,1,2,\dots$), the largest value of m being $j/2$ if j is even and $(j-1)/2$ if j is odd. [$1 = (-1)^{1/2}$]

2. The $y_3^{(j)}$ and $y_4^{(j)}$ are sums of terms of the form $\sin (2m+1)\alpha\tau$ and $\cos (2m+1)\alpha\tau$, m having the same properties as above. The $y_3^{(0)}$ and $y_4^{(0)}$ are identically zero.

3. The $y_1^{(j)}$ are the same functions of $(i\tau)$ as the $y_2^{(j)}$ are of $(-i\tau)$ and conversely.

4. The coefficient of $\cos (2m+1)\alpha\tau$ in $y_3^{(j)}$ is the same as the coefficient of $\cos (2m+1)\alpha\tau$ in $y_4^{(j)}$, while the coefficient of $\sin (2m+1)\alpha\tau$ in $y_3^{(j)}$ is the negative of the corresponding coefficient in $y_4^{(j)}$.

5. The $u_3^{(j)}$ and $v_3^{(j)}$ are sums of cosines of even multiples of $\alpha\tau$ the highest multiple being $j+1$ if j is odd and j if j is even.

It is desired to show that they can be computed for $j = n$, and that they have the same form. The general term for the solution (6:2) will then follow immediately.

The differential equations for the n -th order coefficients in equations (6:1) are :

$$\begin{aligned}
 \frac{dy_1^{(n)}}{d\tau} &= \alpha i y_1^{(n)} + \alpha i \sum_{j=1}^n \delta_j y_1^{(n-j)} + \frac{R_2^{(n-1)} - R_1^{(n-1)}}{2} \\
 &\quad + \sum_{j=1}^{n-1} \delta_j \frac{(R_2^{(n-1-j)})_1 - R_1^{(n-1-j)}}{2}, \\
 (6:15') \quad \frac{dy_2^{(n)}}{d\tau} &= -\alpha i y_2^{(n)} - \alpha i \sum_{j=1}^n \delta_j y_2^{(n-j)} - \frac{(R_2^{(n-1)})_1 + R_1^{(n-1)}}{2} \\
 &\quad - \sum_{j=1}^{n-1} \delta_j \frac{(R_2^{(n-1-j)})_1 + R_1^{(n-1-j)}}{2},
 \end{aligned}$$

$$\frac{dy_3^{(n)}}{d\tau} = \rho y_3^{(n)} + \rho \sum_{j=1}^n \delta_j y_3^{(n-j)} + \frac{R_3^{(n-1)} + R_4^{(n-1)}}{2} + \sum_{j=1}^{n-1} \delta_j \frac{(R_3^{(n-1-j)} + R_4^{(n-1-j)})}{2},$$

$$\frac{dy_4^{(n)}}{d\tau} = -\rho y_4^{(n)} - \rho \sum_{j=1}^n \delta_j y_4^{(n-j)} + \frac{R_3^{(n-1)} - R_4^{(n-1)}}{2} + \sum_{j=1}^{n-1} \delta_j \frac{(R_3^{(n-1-j)} - R_4^{(n-1-j)})}{2},$$

(6:15")

$$\frac{du_3^{(n)}}{d\tau} = -m_3 \sum_{s,t} u_1^{(s)} u_2^{(t)} - m_3 \sum_{p=1}^n \delta_p \left(\sum_{s,t} u_1^{(s)} u_2^{(t)} \right),$$

$$s + t = n - 1$$

$$s + t = n - 1 - p$$

$$\frac{dv_3^{(n)}}{d\tau} = \sum_{s,t} (v_1^{(s)} u_2^{(t)} - v_2^{(s)} u_1^{(t)}) + \sum_{p=1}^{(n-1)} \delta_p \left(\sum_{s,t} v_1^{(s)} u_2^{(t)} - v_2^{(s)} u_1^{(t)} \right),$$

$$s + t = n - 1$$

$$s + t = n - 1 - p$$

where $R_1^{(n-1)}$ is the coefficient of ϵ^{n-1} when (6:1) is substituted in R_1 , etc.

A necessary condition that the solution of equations (6:15) be periodic is that there exist a δ_n which makes the coefficient of $e^{\epsilon 1\tau}$ vanish in the first equation and at the same time makes the coefficient of $e^{-\epsilon 1\tau}$ in the second equation, vanish. In view of properties 3 and 4 above and the form of the R 's in (4:7) it follows that the first equation of (6:15) is derived from the second by replacing τ by $-\tau$. Hence a value of δ_n which makes the coefficient of $e^{\epsilon 1\tau}$ in the first equation, vanish, also makes the coefficient of $e^{-\epsilon 1\tau}$ in the second equation, vanish.

Since the arbitrary constants appearing in the solution of the first two equations of (6:15) must be the same by virtue

of the condition, $v_1^{(n)} = v_2^{(n)} = 0$ at $\tau = 0$, it follows from the interchangeability described above that the solutions of the first two equations (6:15) must also satisfy property 3. Hence property 3 is satisfied for $j = n$. Since the arbitrary constant in the solutions of the third and fourth equations of (6:15) must be zero if the solutions are periodic, it follows similarly that these solutions satisfy property 4. Hence this property holds for $j = n$.

In order to establish properties 1 and 2 it will be convenient to write out the expressions for $R_1^{(n-1)}$ and $R_2^{(n-1)}$. The $R_1^{(n-1-p)}$ ($p \neq 0$) and $R_2^{(n-1-p)}$ contain no terms which are not in $R_1^{(n-1)}$, $R_2^{(n-1)}$. Since the constant multipliers will be neglected in what is to follow, the former terms may be omitted. By (4:7) the expressions for $R_1^{(n-1)}$ and $R_2^{(n-1)}$ are:

$$\begin{aligned}
 R_1^{(n-1)} &= \sum_{s,t} (p_1 u_3^{(s)} + q_1 v_3^{(s)}) (y_1^{(t)} - y_2^{(t)}) \\
 &\quad s + t = n - 1 \quad + \sum_{s,t} (r_1 u_3^{(s)} + s_1 v_3^{(s)}) (y_3^{(t)} - y_4^{(t)}), \\
 R_2^{(n-1)} &= \sum_{s,t} (p_2 u_3^{(s)} - q_2 v_3^{(s)}) (y_1^{(t)} + y_2^{(t)}) \\
 &\quad + \sum_{s,t} (r_2 u_3^{(s)} - s_2 v_3^{(s)}) (y_3^{(t)} + y_4^{(t)}).
 \end{aligned}$$

Neglecting the numerical coefficients and making use of properties (1 . . . 4) for $j \leq n - 1$, these expressions may be put in the following form:

$$\begin{aligned}
 R_1^{(n-1)} &\sim \sum_{\bar{k}, \bar{m}} \sum_{k, m} \left(\frac{e^{2k\alpha i \tau} + e^{-2k\alpha i \tau}}{2} \right) \left(\frac{e^{(2m+1)\alpha i \tau} - e^{-(2m+1)\alpha i \tau}}{2i} \right), \\
 (6:16) \quad R_2^{(n-1)} &\sim \sum_{\bar{k}, \bar{m}} \sum_{k, m} \left(\frac{e^{2k\alpha i \tau} + e^{-2k\alpha i \tau}}{2} \right) \left(\frac{e^{(2m+1)\alpha i \tau} + e^{-(2m+1)\alpha i \tau}}{2i} \right),
 \end{aligned}$$

where in the first sum k and m range independently from zero to $\bar{k}$ and $\bar{m}$ respectively and $\bar{k}$ and $\bar{m}$ in the second sum satisfy the

relation

$$\begin{aligned}\bar{k} + \bar{m} &= (n - 1)/2 \text{ for } n \text{ odd,} \\ \bar{k} + \bar{m} &= n/2 \quad \text{for } n \text{ even.}\end{aligned}$$

The sign ($\sim$) indicates that (6:16) are not true equations but merely give the form of the equations as functions of τ omitting the constant coefficients. The $R_3^{(n-1)}$ and $R_4^{(n-1)}$ have exactly the same form except that in this case it is convenient to express the result in terms of sines and cosines. The values are

$$\begin{aligned}(6:17) \quad R_3^{(n-1)} &\sim \sum_{\bar{k}, \bar{m}} \frac{\bar{k}, \bar{m}}{k, m} \frac{\sin(2k+2m+1)\alpha\tau - \sin(2k-2m-1)\alpha\tau}{2}, \\ R_4^{(n-1)} &\sim \sum_{\bar{k}, \bar{m}} \frac{\bar{k}, \bar{m}}{k, m} \frac{\cos(2k+2m+1)\alpha\tau + \cos(2k-2m-1)\alpha\tau}{2}.\end{aligned}$$

From the form of (6:16) and (6:17) and properties 3 and 4 which have already been established for $j = n$, the solution of the first four equations of (6:15) have the form

$$\begin{aligned}y_1^{(n)} &= K_1^{(n)} e^{-\alpha\tau} + a_1^{(n)} e^{-\alpha\tau} + a_3^{(n)} e^{3\alpha\tau} + a_4^{(n)} e^{-3\alpha\tau} + \dots \\ &\quad + a_{\chi}^{(n)} e^{(2\chi+1)\alpha\tau} + a_{\chi+1}^{(n)} e^{-(2\chi+1)\alpha\tau}, \\ y_2^{(n)} &= K_1^{(n)} e^{-\alpha\tau} + a_1^{(n)} e^{\alpha\tau} + a_3^{(n)} e^{-3\alpha\tau} + a_4^{(n)} e^{3\alpha\tau} + \dots \\ &\quad + a_{\chi}^{(n)} e^{-(2\chi+1)\alpha\tau} + a_{\chi+1}^{(n)} e^{(2\chi+1)\alpha\tau}, \\ (6:18) \quad y_3^{(n)} &= \sum_{p=0}^{\chi} c_p \cos(2p+1)\alpha\tau + \sum_{p=0}^{\chi} d_p \sin(2p+1)\alpha\tau, \\ y_4^{(n)} &= \sum_{p=0}^{\chi} c_p \cos(2p+1)\alpha\tau - \sum_{p=0}^{\chi} d_p \sin(2p+1)\alpha\tau, \\ \chi &= (n - 1)/2 \text{ if } n \text{ is odd,} \\ \chi &= n/2 \quad \text{if } n \text{ is even.}\end{aligned}$$

The coefficients in (6:18) are determined by the usual method.

Hence properties 1 and 2 are established for $j = n$.

It remains to establish property 5. Since the last two equations of (6:15) have the same functions of τ (except for the constant coefficients) in their right members, it will suffice to treat the equation involving $u_3^{(n)}$. By properties (1 . . . 4) and transformation (4:5) it is readily verified that the equation of (6:15) involving $u_3^{(n)}$ is of the form

$$\begin{aligned} \frac{du_3^{(n)}}{d\tau} &\sim - \sum_{\bar{k}, \bar{m}} \sum_{k, m}^{\bar{k}, \bar{m}} [\cos (2k+1)\angle\tau \sin (2m+1)\angle\tau] , \\ &\sim - \sum_{\bar{k}, \bar{m}} \sum_{k, m}^{\bar{k}, \bar{m}} [\sin 2(k+m+1)\angle\tau - \sin 2(k-m)\angle\tau] , \\ \bar{k} + \bar{m} &= n/2 - 1 \text{ for } n \text{ even,} \\ \bar{k} + \bar{m} &= (n - 1)/2 \text{ for } n \text{ odd.} \end{aligned}$$

The integral of this expression is of the form

$$u_3^{(n)} \sim \sum_{\bar{k}, \bar{m}} \sum_{k, m}^{\bar{k}, \bar{m}} \cos 2(k+m+1)\angle\tau - \cos 2(k-m)\angle\tau.$$

This expression, in view of the restrictions on $\bar{k}$ and $\bar{m}$ establishes property 5 for $u_3^{(n)}$. A similar discussion establishes the property for $v_3^{(n)}$, completing the induction proof.

The n -th degree terms of the solution (6:2) are obtainable through transformation (4:5) and will be real valued by properties 1, 2, and 3.

7. Determination of the angles of Euler. The solution of equations (1:5) is readily obtained by making use of the identities (2:1) and the coefficients of the solution (6:2) given by equations (6:3), (6:9), and (6:14). The solution up to third degree terms in ϵ is given by the following equations:

$$\omega_1 = - [(nh_1 + \alpha)a \sin \alpha \tau] \epsilon + (A_{11}^{(1)} \sin \alpha \tau) \epsilon^2 + (A_{11}^{(2)} \sin \alpha \tau + A_{13}^{(2)} \sin 3\alpha \tau) \epsilon^3,$$

$$\omega_2 = [(n + \alpha h_1)a \cos \alpha \tau] \epsilon + (A_{21}^{(1)} \cos \alpha \tau) \epsilon^2 + (A_{21}^{(2)} \cos \alpha \tau + A_{23}^{(2)} \cos 3\alpha \tau) \epsilon^3,$$

$$(7:1) \quad \omega_3 = n + b \epsilon + (A_{32}^{(1)} \cos 2\alpha \tau) \epsilon^2 + (A_{32}^{(2)} \cos 2\alpha \tau) \epsilon^3,$$

$$\gamma_1 = - (h_1 a \sin \alpha \tau) \epsilon + (B_{11}^{(1)} \sin \alpha \tau) \epsilon^2 + (B_{11}^{(2)} \sin \alpha \tau + B_{13}^{(2)} \sin 3\alpha \tau) \epsilon^3,$$

$$\gamma_2 = (a \cos \alpha \tau) \epsilon + B_{21}^{(2)} (\cos \alpha \tau - \cos 3\alpha \tau) \epsilon^3,$$

$$\gamma_3 = 1 + c \epsilon + (B_{32}^{(1)} \cos 2\alpha \tau) \epsilon^2 + (B_{32}^{(2)} \cos 2\alpha \tau) \epsilon^3.$$

In the solution of equations (1:2) for the derivatives of the Eulerian angles, the value of (ψ') is found, after taking equations (1:1) into account, to be the following:

$$(7:2) \quad \psi' = \frac{\omega_1 \gamma_1 + \omega_2 \gamma_2}{\sin^2 \Theta}.$$

Assuming that $c < 0$ in (7:1) the factor $1/(\sin^2 \Theta)$ is readily computed by means of the last equation of (7:1) and the relation $\sin^2 \Theta = 1 - \gamma_3^2$. When the values from (7:1) are substituted in the numerator of (7:2), it has the value

$$\omega_1 \gamma_1 + \omega_2 \gamma_2 = (H_1 + H_2 \cos 2\alpha \tau) \epsilon^2 + (H_3 + H_4 \cos 2\alpha \tau) \epsilon^3 + (H_5 + H_6 \cos 2\alpha \tau + H_7 \cos 4\alpha \tau) \epsilon^4,$$

whence the value of (7:2) is¹ obtainable as a function of τ by

¹In these formulae $c < 0$, and can be determined as a unique power series in ϵ so that the sum of the squares of γ_1 , γ_2 , and γ_3 is unity.

multiplying the above expression by $\frac{1}{\sin^2 \theta}$. It should be remembered that in any numerical work, c should be determined as a power series in ϵ so that the sum of the squares of γ_1 , γ_2 , and γ_3 , in (7:1) is unity. After this has been done, it will be found that $\sin^2 \theta$ contains a factor ϵ^2 which cancels with the factor ϵ^2 in the numerator of (7:2) and hence the expansion of ψ' in powers of ϵ begins with a term independent of ϵ .

8. Case 2, all of the characteristic roots, pure imaginary. It will be shown that the periodic solutions for the case under consideration are of the same type as those considered in section 6 and have the same restrictions on their initial conditions. Hence if at least two of the roots of the characteristic equation are imaginary, the periodic solutions are of the same type whether the motion is stable or unstable.

The solution of equations (5:15) is of the form of (6:1), the coefficients being determined by substituting (6:1) in the differential equations (5:15) and equating coefficients of like powers of ϵ . It was found in section 5 that a necessary condition for the existence of periodic solutions was that the arbitrary constants in (5:12) have the values

$$\bar{c}_2^{(1)} = \bar{c}_2^{(2)} = \bar{b}_2/2, \bar{c}_2^{(3)} = \bar{c}_2^{(4)} = 0$$

For these constants, the terms independent of ϵ in the solution (6:1) of equations (5:15) are

$$\begin{aligned}
 (8:1) \quad y_1^{(0)} &= (\bar{b}_2/2)e^{\bar{\alpha}1\tau}, & y_3^{(0)} &= 0, \\
 y_2^{(0)} &= (\bar{b}_2/2)e^{-\bar{\alpha}1\tau}, & y_4^{(0)} &= 0, \\
 u_3^{(0)} &= \bar{b}, & v_3^{(0)} &= \bar{c}.
 \end{aligned}$$

The differential equations for the first degree terms in the solution (6:1) are obtained in the usual manner and have the following form:

$$\begin{aligned}
 \frac{dy_1^{(1)}}{d\tau} &= \bar{\alpha}1y_1^{(1)} + \left[\bar{s}_1\bar{\alpha} + \frac{\bar{p}_2\bar{b}-\bar{q}_2\bar{c}}{2} - \frac{\bar{p}_1\bar{b}+\bar{q}_1\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{\bar{\alpha}1\tau} \\
 &\quad + \left[\frac{\bar{p}_2\bar{b}-\bar{q}_2\bar{c}}{2} + \frac{\bar{p}_1\bar{b}+\bar{q}_1\bar{c}}{2} \right] \bar{c}_2^{(1)} e^{-\bar{\alpha}1\tau}, \\
 \frac{dy_2^{(1)}}{d\tau} &= -\bar{\alpha}1y_2^{(1)} - \left[\bar{s}_1\bar{\alpha} + \frac{\bar{p}_2\bar{b}-\bar{q}_2\bar{c}}{2} - \frac{\bar{p}_1\bar{b}+\bar{q}_1\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{-\bar{\alpha}1\tau} \\
 &\quad - \left[\frac{\bar{p}_2\bar{b}-\bar{q}_2\bar{c}}{2} + \frac{\bar{p}_1\bar{b}+\bar{q}_1\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{\bar{\alpha}1\tau}, \\
 (8:2) \quad \frac{dy_3^{(1)}}{d\tau} &= \bar{\beta}1y_3^{(1)} + \left[\frac{\bar{p}_4\bar{b}-\bar{q}_4\bar{c}}{2} - \frac{\bar{p}_3\bar{b}+\bar{q}_3\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{\bar{\alpha}1\tau} \\
 &\quad + \left[\frac{\bar{p}_4\bar{b}-\bar{q}_4\bar{c}}{2} + \frac{\bar{p}_3\bar{b}+\bar{q}_3\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{-\bar{\alpha}1\tau}, \\
 \frac{dy_4^{(1)}}{d\tau} &= -\bar{\beta}1y_4^{(1)} - \left[\frac{\bar{p}_4\bar{b}-\bar{q}_4\bar{c}}{2} + \frac{\bar{p}_3\bar{b}+\bar{q}_3\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{\bar{\alpha}1\tau} \\
 &\quad - \left[\frac{\bar{p}_4\bar{b}-\bar{q}_4\bar{c}}{2} - \frac{\bar{p}_3\bar{b}+\bar{q}_3\bar{c}}{2} \right] i\bar{c}_2^{(1)} e^{-\bar{\alpha}1\tau}, \\
 \frac{du_3^{(1)}}{d\tau} &= -m_3u_1^{(0)}u_2^{(0)}, & \frac{dv_3^{(1)}}{d\tau} &= (v_1^{(0)}u_2^{(0)} - v_2^{(0)}u_1^{(0)}).
 \end{aligned}$$

In order that the solution of equations (8:2) be periodic with period $(2\pi p/\alpha) = (2\pi q/\beta)$ it is necessary and sufficient that $\bar{s}_1$ have the value

$$\mathcal{J}_1 = [(\bar{p}_1 - \bar{p}_2)\bar{b} + (\bar{q}_1 + \bar{q}_2)\bar{c}]/2\bar{\alpha}.$$

The solution of equations (8:2) is of the form

$$\begin{aligned} y_1^{(1)} &= k_1^{(1)} e^{\bar{\alpha}1\tau} + \bar{a}_1^{(1)} e^{-\bar{\alpha}1\tau}, \quad y_3^{(1)} = k_3^{(1)} e^{\bar{\beta}1\tau} + \bar{a}_3^{(1)} e^{-\bar{\alpha}1\tau} + \bar{b}_3^{(1)} e^{\bar{\alpha}1\tau}, \\ (8:3) \quad y_2^{(1)} &= k_2^{(1)} e^{-\bar{\alpha}1\tau} + \bar{b}_2^{(1)} e^{\bar{\alpha}1\tau}, \quad y_4^{(1)} = k_4^{(1)} e^{-\bar{\beta}1\tau} + \bar{a}_4^{(1)} e^{-\bar{\alpha}1\tau} + \bar{b}_4^{(1)} e^{\bar{\alpha}1\tau}, \\ u_3^{(1)} &= \bar{A}_{32}^{(1)} \cos 2\bar{\alpha}\tau, \quad v_3^{(1)} = \bar{B}_{32}^{(1)} \cos 2\bar{\alpha}\tau. \end{aligned}$$

The $k_1^{(1)}$, $k_2^{(1)}$, $k_3^{(1)}$ and $k_4^{(1)}$ are left undetermined for the present while the remainder of the coefficients are determined in the usual manner.

The coefficients of the second degree terms in the solution (6:1), of the differential equations (5:15), are determined in the same manner as in section 6 up to the solution of the equations corresponding to (6:10). In these equations however, there are terms of period $2\pi/\bar{\beta}$ due to the terms $k_3^{(1)} e^{\bar{\beta}1\tau}$ and $k_4^{(1)} e^{-\bar{\beta}1\tau}$, appearing in (8:3). The $\mathcal{J}_2$ is determined as in (6:10) to eliminate the non-periodic terms in the solution of the first two equations. The non-periodic terms in the solution of the third and fourth equations due to the terms $k_3^{(1)} e^{\bar{\beta}1\tau}$ and $k_4^{(1)} e^{-\bar{\beta}1\tau}$ are eliminated if and only if $k_3^{(1)} = k_4^{(1)} = 0$. The same argument is valid at each step of the process, for at each step the terms in $e^{\bar{\beta}1\tau}$ and $e^{-\bar{\beta}1\tau}$ will appear, in the differential equations corresponding to (6:10), with arbitrary constants as factors. The non-periodic terms of the solution will be eliminated only by setting these arbitrary constants equal to zero. Hence each term of the solution (6:1) is of period $2\pi/\bar{\alpha}$ and is free of the terms of period $2\pi/\bar{\beta}$. The $k_1^{(1)}$ and $k_2^{(1)}$ can now be determined as in section 6 by the condition $v_1^{(1)} = v_2^{(1)} = 0$ at $\tau = 0$, and transformation (5:14). The relations between the coefficients

are found to be

$$k_1^{(1)} = k_2^{(1)} = -(\bar{a}_1^{(1)} + \bar{a}_3^{(1)} + \bar{b}_3^{(1)}), \quad \bar{a}_1^{(1)} = \bar{b}_2^{(1)}, \quad \bar{a}_3^{(1)} = \bar{b}_4^{(1)}, \quad \bar{b}_5^{(1)} = \bar{a}_4^{(1)}.$$

It should be noticed however that there is another periodic solution with period $2\pi/\beta$ which is free of the terms of period $2\pi/\alpha$, for if the roots of the characteristic equation are all pure imaginaries, either pair of roots may be selected for the construction of the periodic solution.

9. Summary of results. The study of the motion of a rigid body with one point fixed, can be reduced to a study of the solutions of the Euler differential equations (1:5). These equations have a special solution $\omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0$, $\omega_3 = n$, $\gamma_3 = 1$ and a similar one with γ_3 negative. A study of the solutions in the neighborhood of the first of these special ones is accomplished by making the transformation (2:1) and reducing equations (1:5) to the form (2:2). They are solvable as power series in the parameter ϵ . The terms of this solution independent of ϵ are obtained by solving equations (2:2) with $\epsilon = 0$. These terms are the coefficients of the first degree terms in ϵ in the solution of equations (1:5), by virtue of the transformation (2:1), and hence they furnish the basis for the discussion of stability given in section 3. Theorem (3:1) states the conditions for stability in the neighborhood of the special solution $\omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0$, $\omega_3 = n$, $\gamma_3 = 1$, and theorem (3:2) gives the corresponding conditions for the solution $\omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0$, $\omega_3 = n$, $\gamma_3 = -1$.

The existence of periodic solutions of Euler's equations (1:5), in the neighborhood of the special solutions mentioned above, is proved in sections 4 and 5. In section 4 the proof is made for the case in which two of the roots of the characteristic

equation are real and two are pure imaginary numbers. Equations (2:2) are reduced to the form (4:6) by means of the transformation (4:5). It is evident from the nature of transformations (2:1) and (4:5), that if the solution of equations (4:6) as a power series in ϵ is periodic, the same will be true of the solution of equations (1:5). The solution of equations (4:6) for $\epsilon = 0$ is periodic with period $2\pi/\alpha$, for special values of the initial conditions, and is given by the terms independent of ϵ in (4:9). These initial conditions are then given certain increments in (4:8), and these increments and δ are so determined as power series in ϵ that the general solution (4:9) is periodic with period $2\pi/\alpha$ in τ . The period in t is obtained by substituting this value of δ in the last of equations (2:1). In section 5 a similar discussion is given for the case in which all of the roots of the characteristic equation are pure imaginary and commensurable. These are the only cases for which periodic solutions of equations (1:5), for the problem under consideration, exist.

In section 6, there is given a direct construction of the periodic solution of equations (4:6) for the case in which two of the characteristic roots are real and two are pure imaginary numbers. Through transformations (4:5) and (2:1) the corresponding solution of equations (1:5) is obtained. It is shown that the general term of the solution is obtainable and that δ can be determined as a power series in ϵ step by step so that the solution is periodic. MacMillan [7] has shown that under these circumstances the δ series or in fact any number of parameters in the differential equations which appear in the solution as power series in an arbitrary parameter, converge. In section 4 this proof is made to depend upon the well known implicit function theorem. In section 7 the solution of equations (1:5)

is used for the determination of the angles of Euler and thus the solution is made available for geometric interpretation. It will be seen on examining the numerical example (sec. 10) that the axis of the top makes small oscillations in the vertical plane through this axis, as this plane rotates about the vertical line through the point of support.

In section 5 it was shown that the term of the solution of equations (2;2) independent of ϵ , has the same restriction on its initial conditions when all of the roots of the characteristic equation are pure imaginary as for the case in which two are real and two are pure imaginary. This fact was used in section 8 to show that the periodic solution of equations (5;15) and hence of (1;5) has only terms involving α or β separately, but does not have a combination of the two, even when the roots of the characteristic equation are all pure imaginary and commensurable.

A physical interpretation of the period P in t corresponding to the period $2\pi/\alpha$ in τ may be obtained by considering a right hand coordinate system with the axis of x directed along the line K (Figure 1) and the axis of z vertical. With respect to this rotating system, equations (7;1) and (1;1) show that after a period of time P has elapsed, the top and the instantaneous axis of rotation retake their initial positions.

10. Numerical example. Consider the motion of a top such that the ratios of C to A and B have the values $3/4$ and $3/2$ respectively. These values lie in the region (cdqp) of Figure 2 (sec. 3). If B has the value unity then the three moments of inertia (A, B, C) have the values $(2, 1, 3/2)$ respectively. Suppose further that the mass of the top and the distance of the center of gravity from the point of support are unity, then if

$n = 10$ (radians per second) the roots of the characteristic equation are

$$\lambda_1 = 7.3021, \quad \lambda_2 = -7.3021, \quad \lambda_3 = 3.719, \quad \lambda_4 = -3.719.$$

Taking $b = 10$ and $a = 1$ in equations (6:4), the constant c of these equations is left arbitrary for the moment and the coefficients in equations (4:7), (6:7), (6:9), (6:10), (6:12), and (6:14), are computed in terms of c in the order given. The constant c is then determined as a power series in ϵ by the condition that the sum of the squares of Y_1 , Y_2 , and Y_3 in equations (7:1) is unity. Since the sum of the squares of the Y 's set equal to a constant is an integral of the differential equations (1:5), it suffices to impose the condition that this constant be unity for a particular value of τ , viz. $\tau = 0$. The value of c in equations (6:4) is thus found to be

$$c = -.4138\epsilon - .1596\epsilon^2 \dots$$

After the coefficients in equations (7:1) have been computed, ψ is determined by integrating equation (7:2). It should be remembered when integrating this equation that the t in the derivative should be replaced by $(1 + S)\tau$. The value of ψ , θ , and P for this special example are

$$\begin{aligned} \psi &= 10\tau - \frac{1}{2}\sin^{-1}\left(\frac{.8093\sin 2\alpha\tau}{.8276 + .1724\cos 2\alpha\tau}\right) \\ (9:1) \quad & - \left\{ 15.17\tau + \frac{.1974\sin 2\alpha\tau}{.8276 + .1724\cos 2\alpha\tau} \right. \\ & \left. - .4668\sin^{-1}\left(\frac{.8093\sin 2\alpha\tau}{.8276 + .1724\cos 2\alpha\tau}\right) \right\}\epsilon, \dots, \\ \cos\theta &= 1 - (.4138 + .08613\cos 2\alpha\tau)\epsilon^2 \\ & - .1596(1 - \cos 2\alpha\tau)\epsilon^3 \dots, \\ P &= .8605(1 - 1.625\epsilon \dots), \end{aligned}$$

where P is the period in t of the solution of equations (1:5) for this special example. The constants of integration are chosen so that ψ vanishes with τ .

The motion of the projection of the point on the axis of the top which lies on the unit sphere, can be represented by the polar coordinates (ρ, ψ) in the horizontal plane, where $\rho = \sin \theta$. The value of $\sin \theta$ could be obtained as a power series in ϵ from the second equation of (9:1), but it is easier to find the numerical values of $\cos \theta$ (as T advances) from (9:1) and to find the corresponding values for $\rho = \sin \theta$ from a trigonometric table.

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VITA

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